Causal Discovery from Observational Data

Brady Neal

causalcourse.com

What if we don’t have the causal graph?

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Causal discovery: data ⟶ causal graph

What if we don't have the causal graph?

Causal discovery: data $\longrightarrow$ causal graph

Structure identification: identifying the causal graph

Independence-Based Causal Discovery

Semi-Parametric Causal Discovery

Faithfulness Assumption

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Recall the Markov assumption: $X \perp\!\!\!\perp_G Y \mid Z \implies X \perp\!\!\!\perp_P Y \mid Z$

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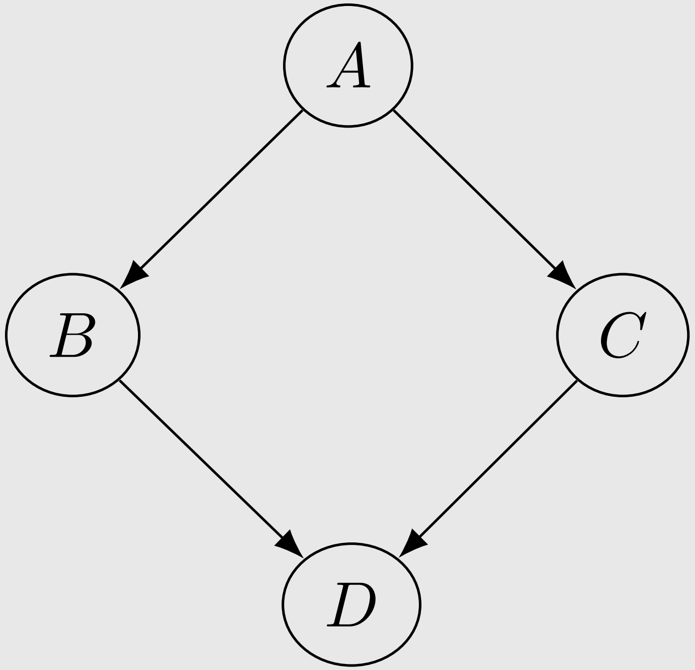
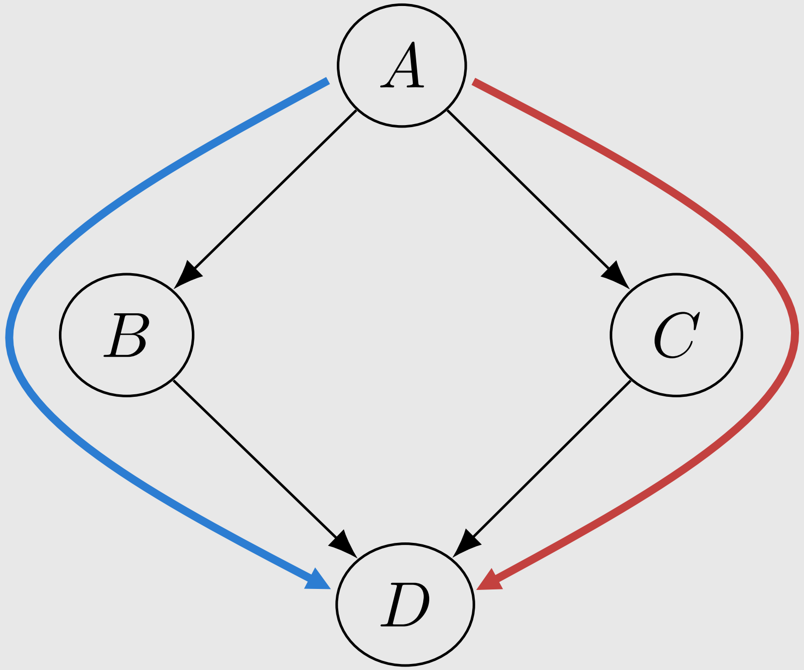


Violation of Faithfulness

Faithfulness: $X \perp\!\!\!\perp_G Y \mid Z \Longleftarrow X \perp\!\!\!\perp_P Y \mid Z$



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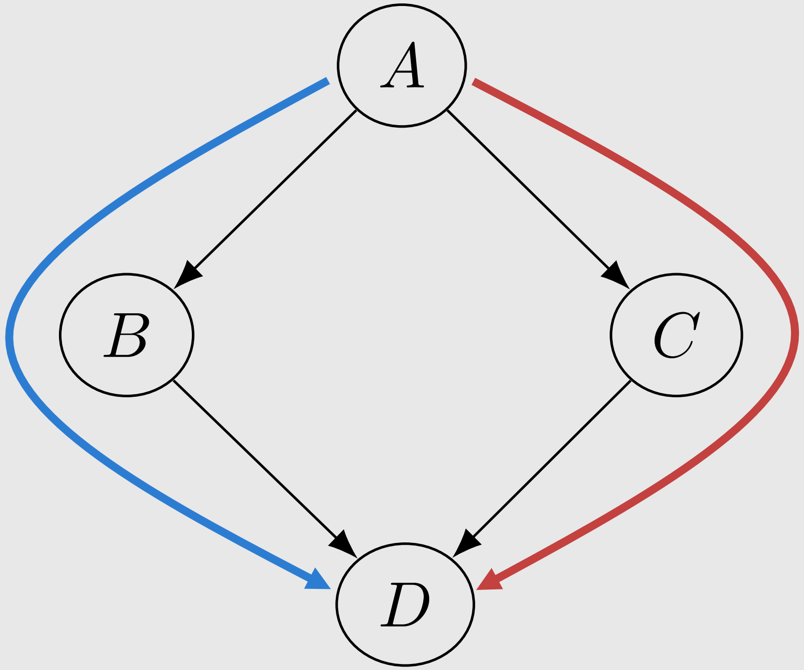

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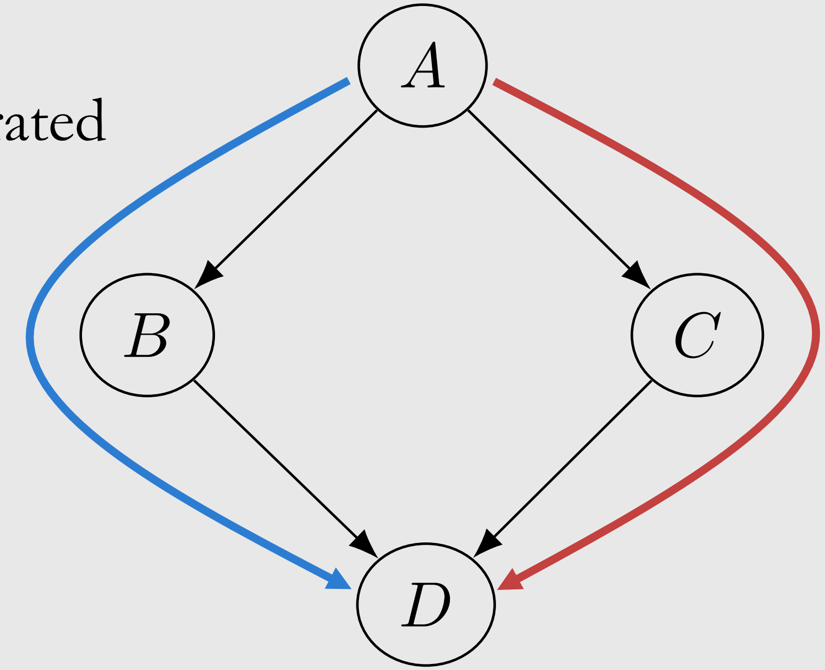
but A and D aren't d-separated



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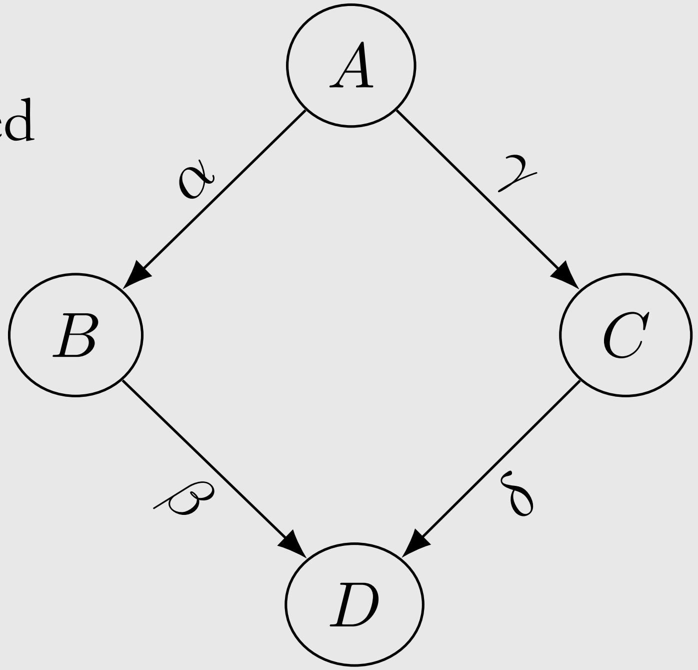

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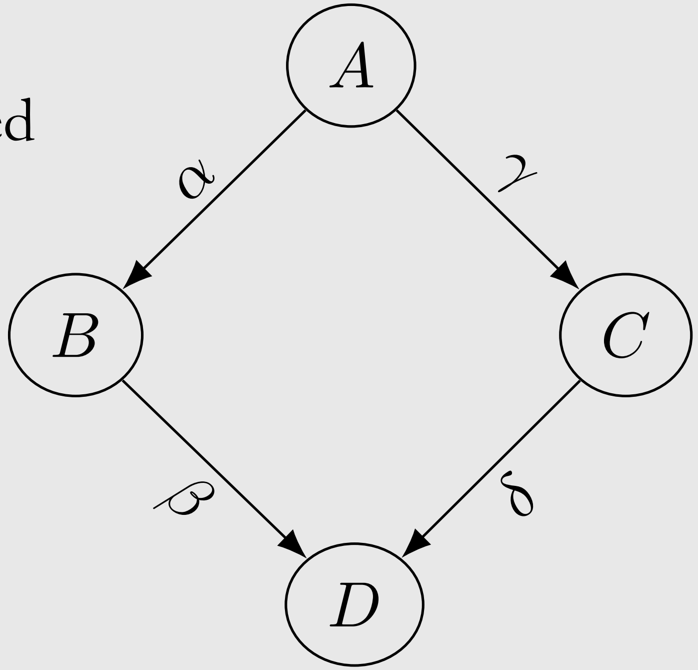

$$B := \alpha A$$
$$C := \gamma A$$
$$D := \beta B + \delta C$$

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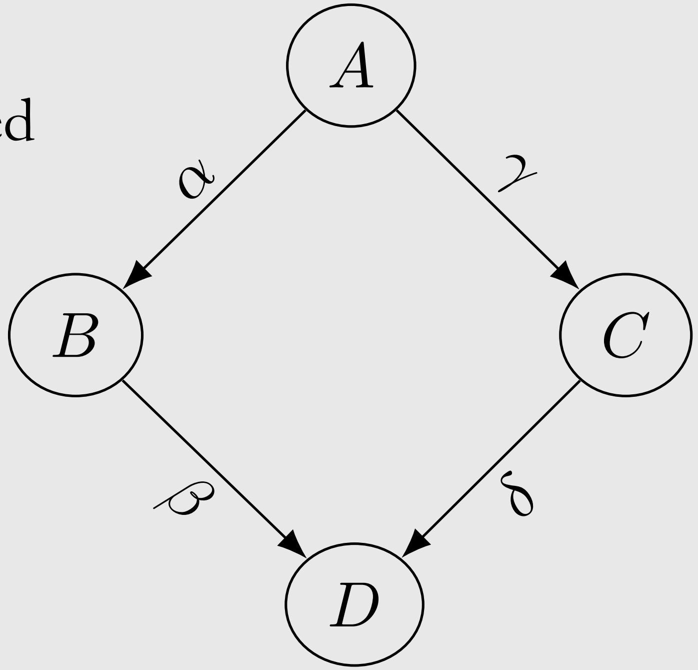

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$$D = (\alpha\beta + \gamma\delta)A$$

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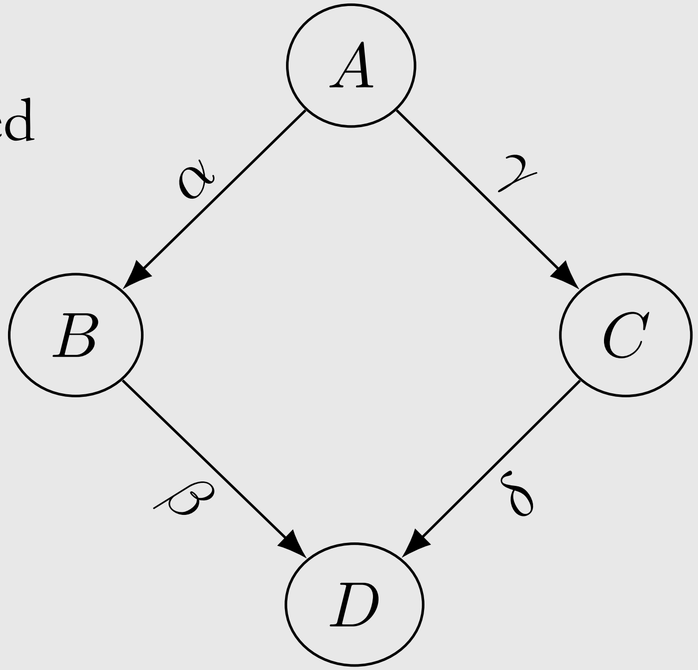

$$B := \alpha A$$
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$$D := \beta B + \delta C$$

$$D = (\alpha\beta + \gamma\delta) A$$

Paths cancel if $\alpha\beta = -\gamma\delta$

Causal Sufficiency and Acyclicity

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All assumptions:

- Markov assumption
- Faithfulness
- Causal sufficiency
- Acyclicity

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All assumptions:

- Markov assumption
- Faithfulness
- Causal sufficiency
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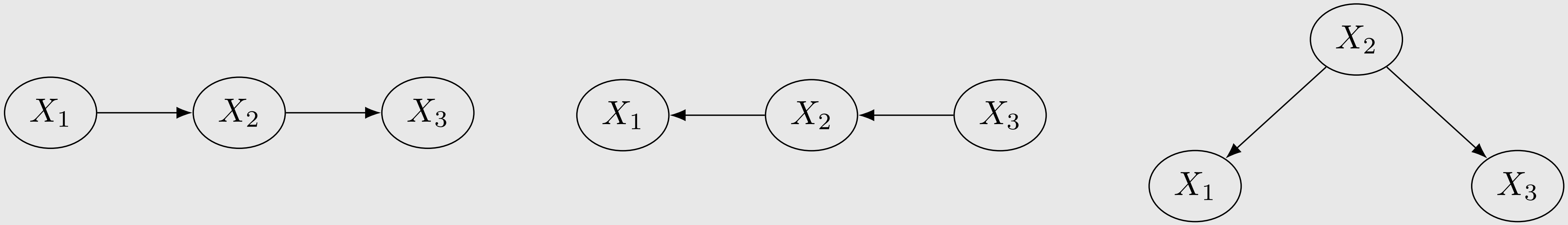
Question:
Why is the Markov assumption (plus causal sufficiency and acyclicity) not enough for learning causal graphs from data?

Chains and Forks Encode Same Independencies



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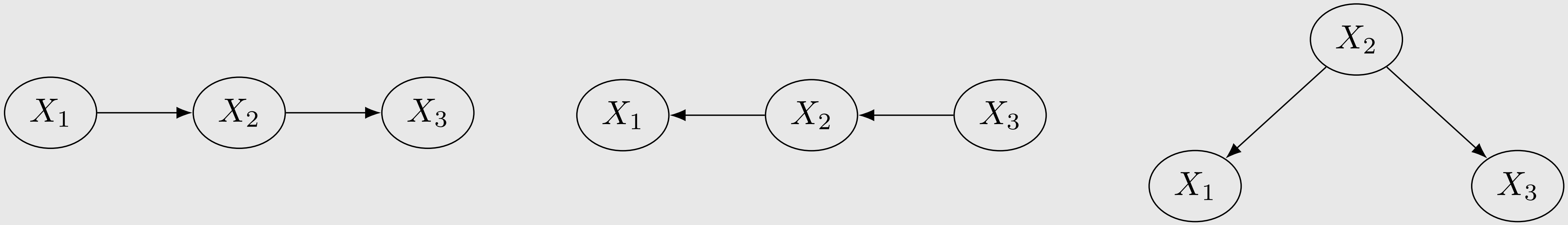

Markov: $X_1 \perp\!\!\!\perp X_3 \mid X_2$

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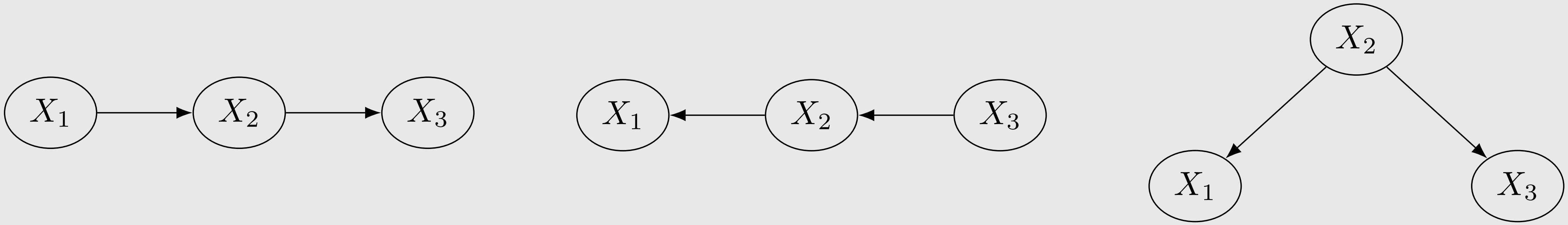

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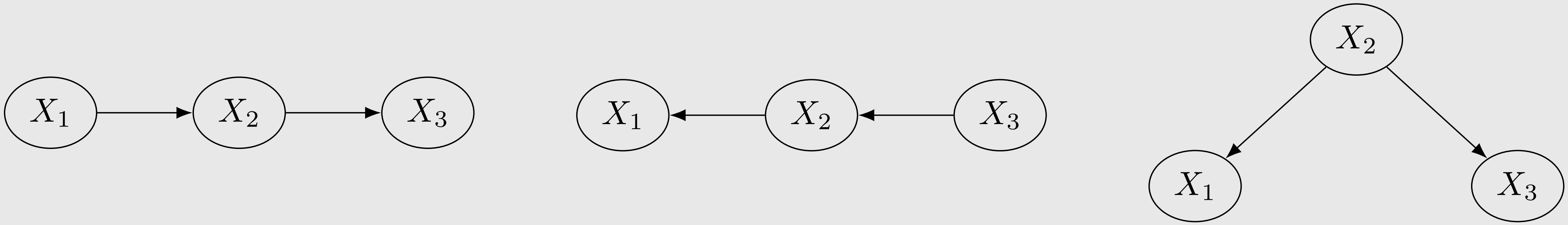

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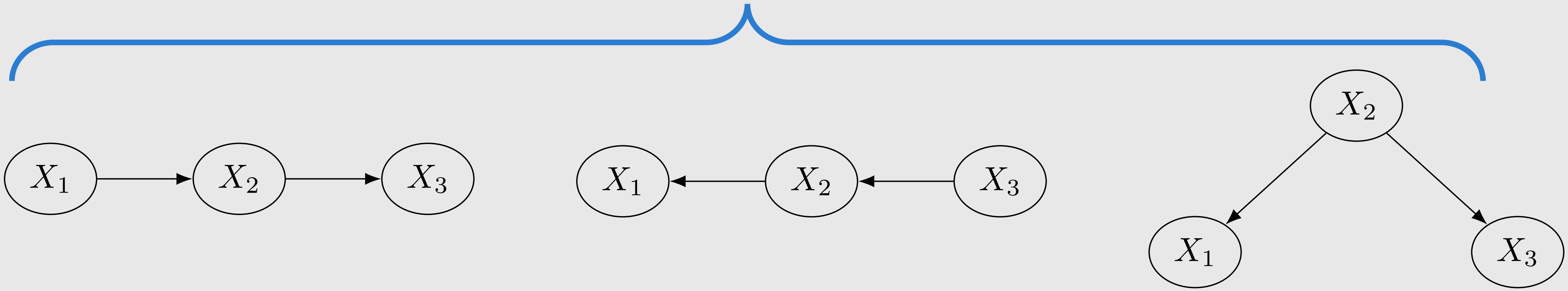
Chains and Forks Encode Same Independencies

Markov equivalent (all in the same Markov equivalence class)



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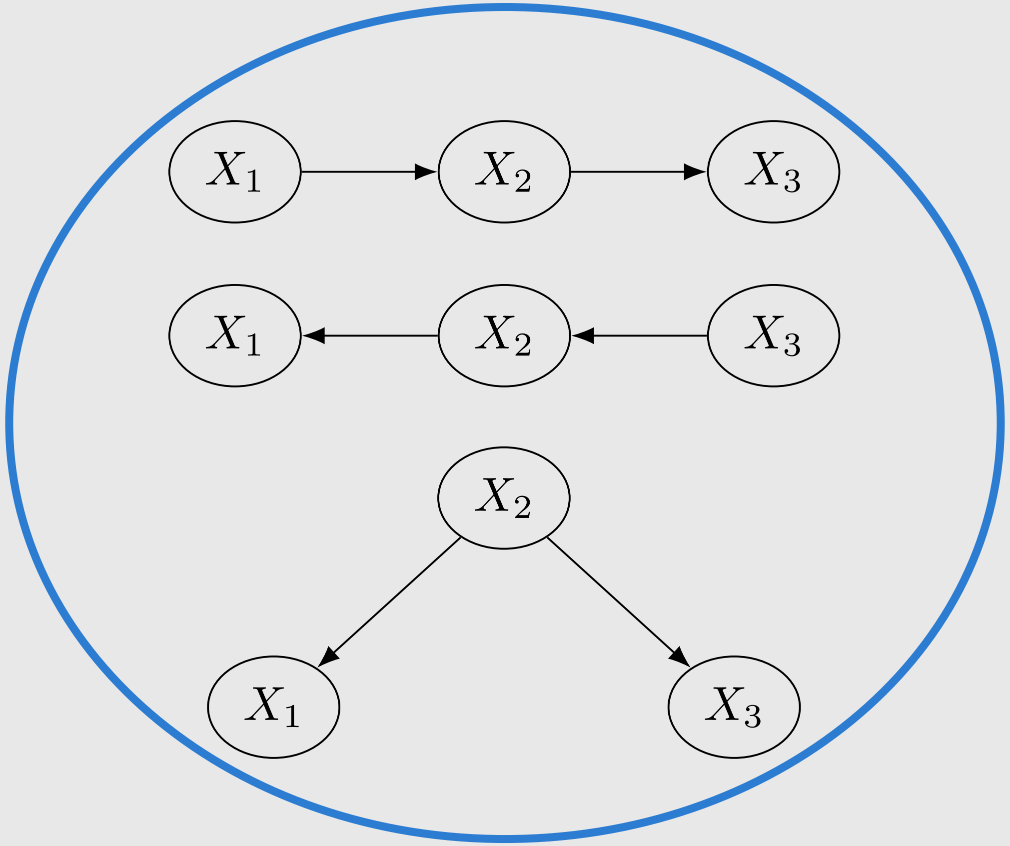
Immoralities are Special

Markov equivalence class where
$X_1 \perp\!\!\!\perp X_3 \mid X_2$ and $X_1 \not\perp\!\!\!\perp X_3$



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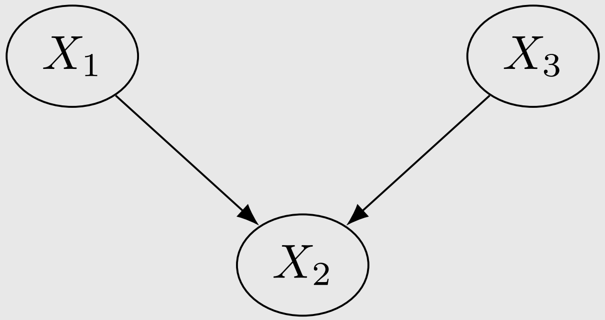

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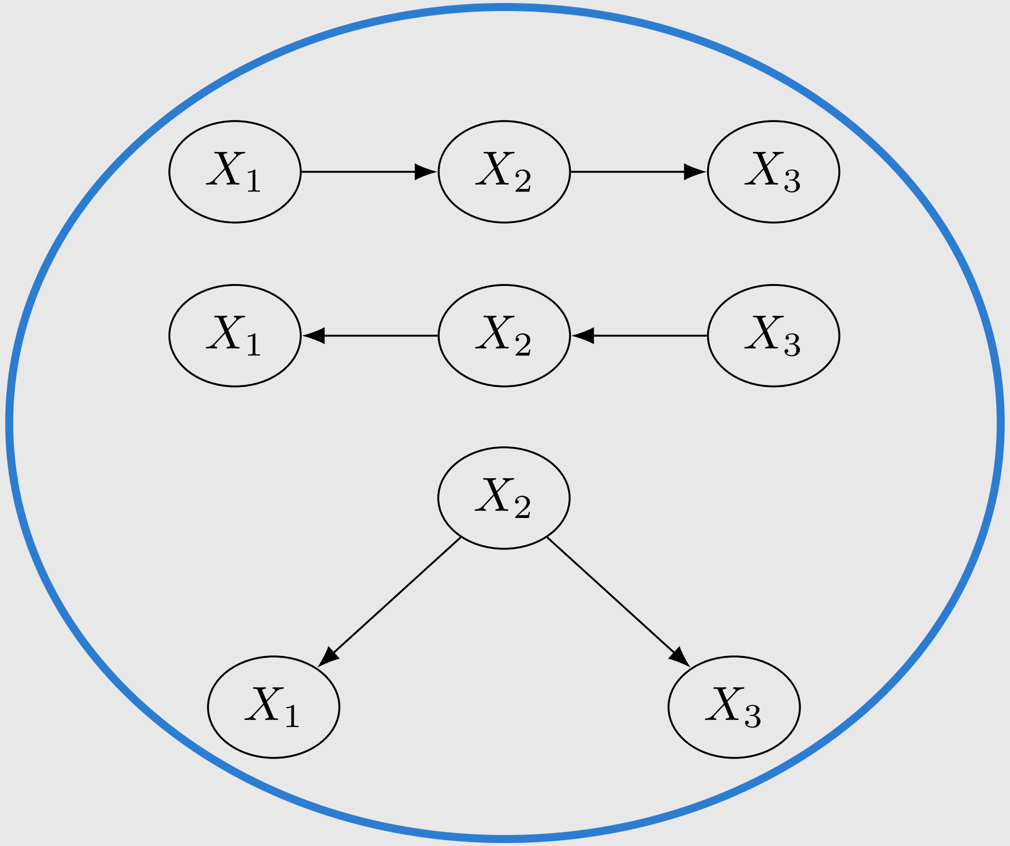

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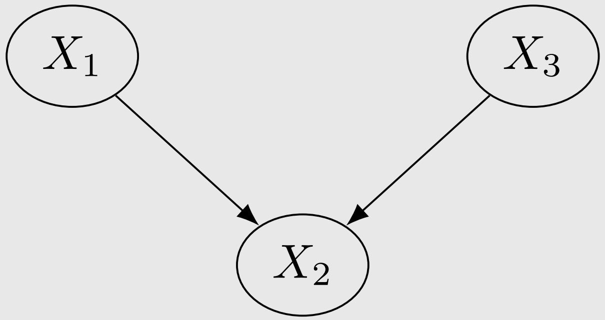

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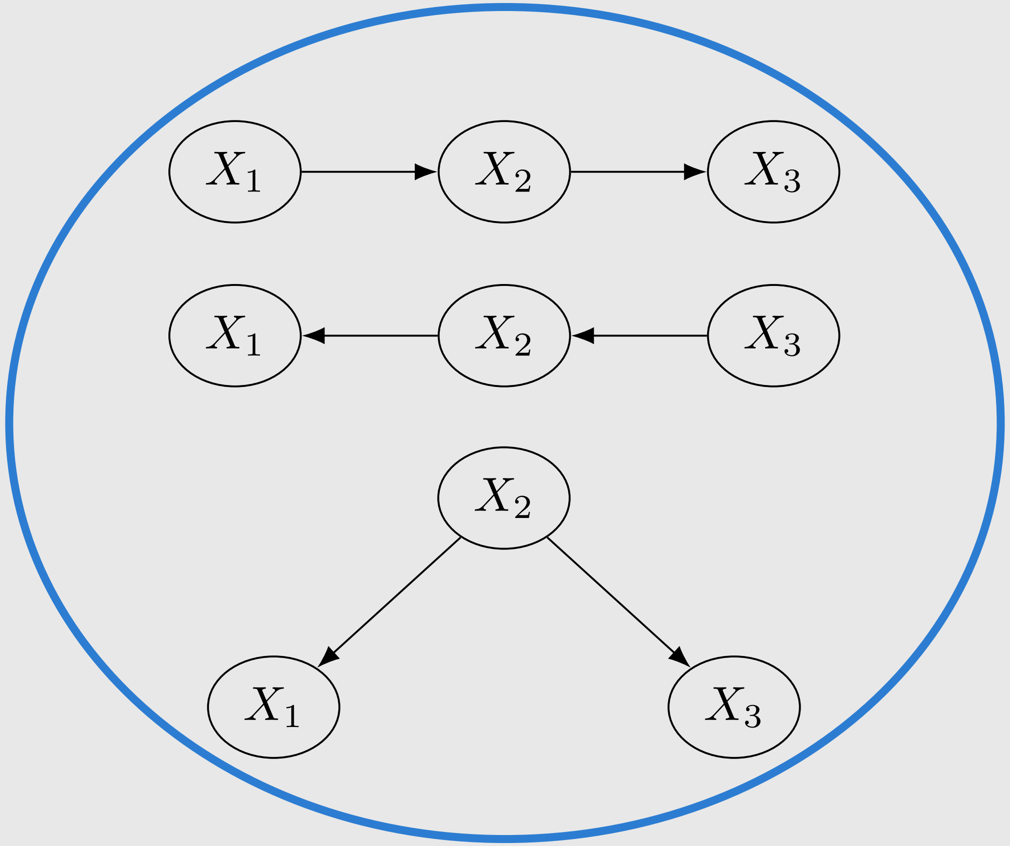

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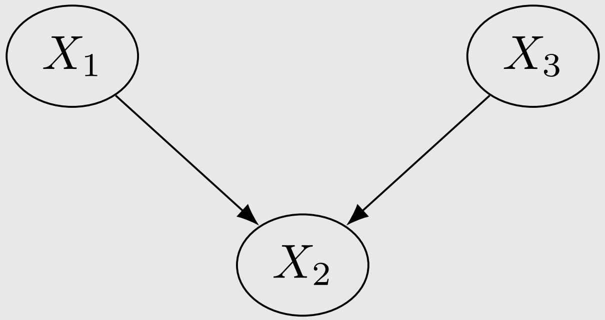

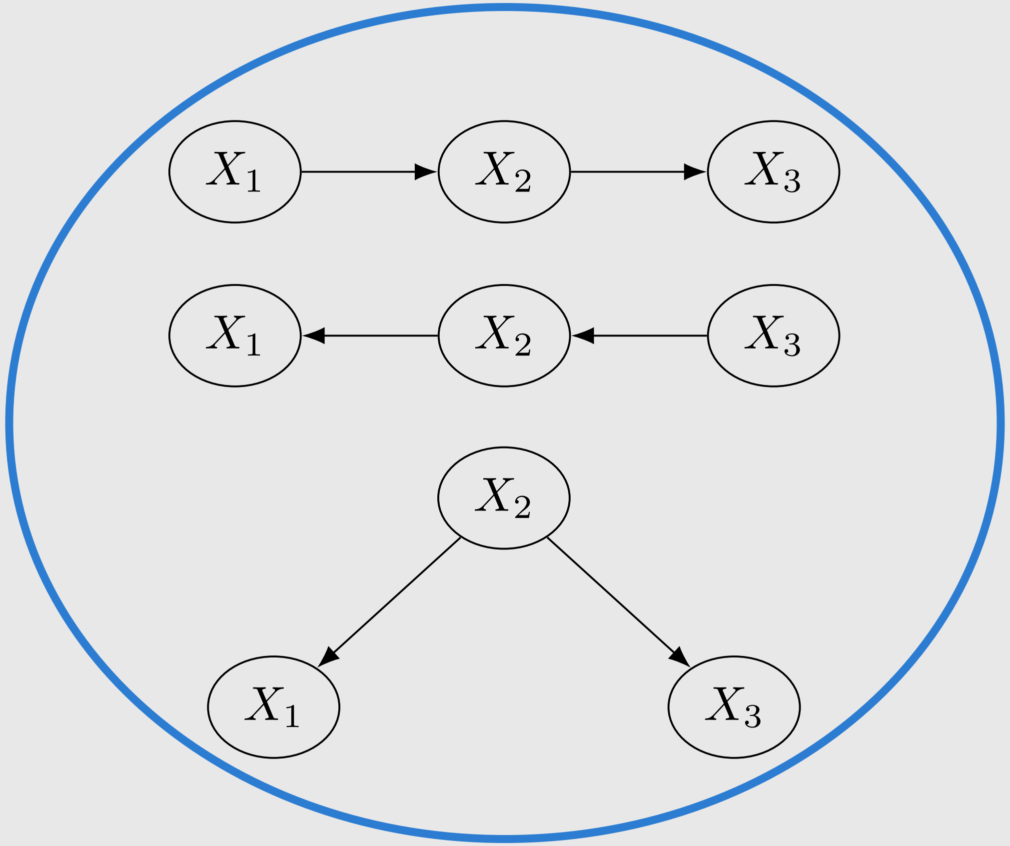

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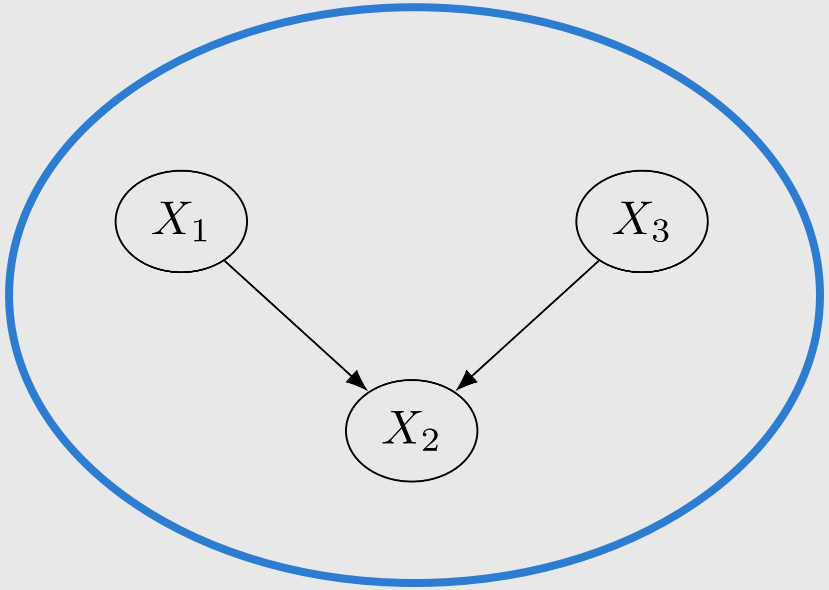

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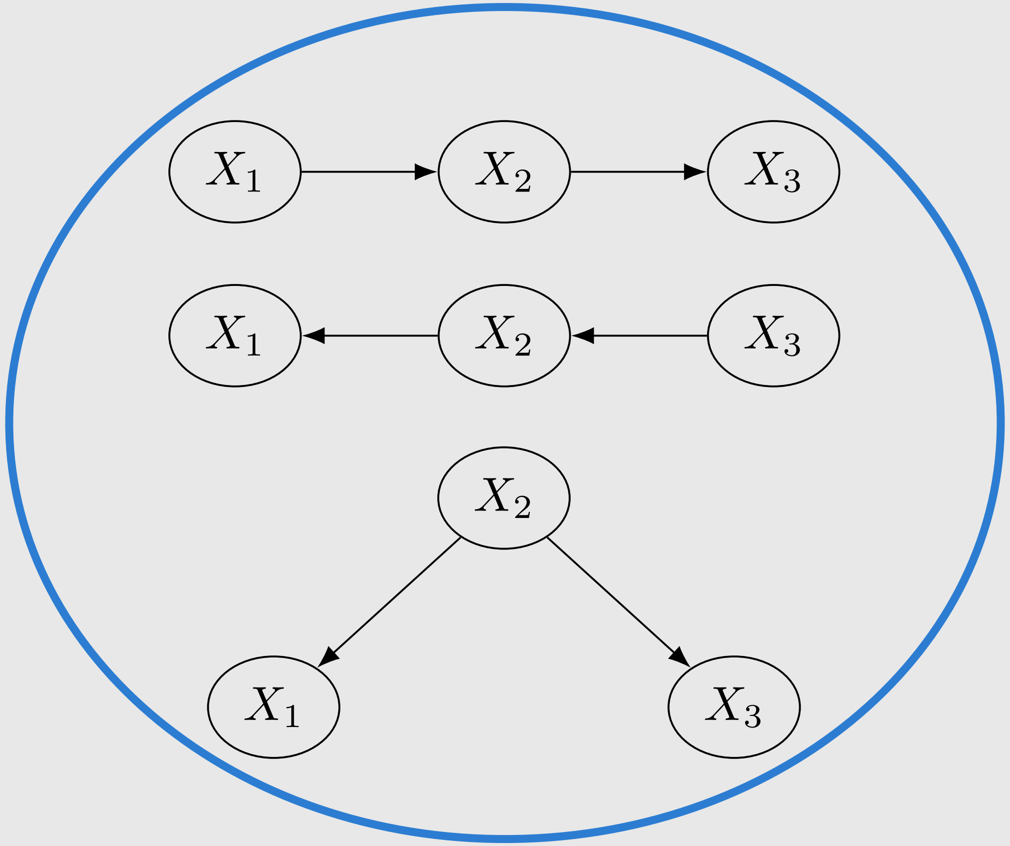
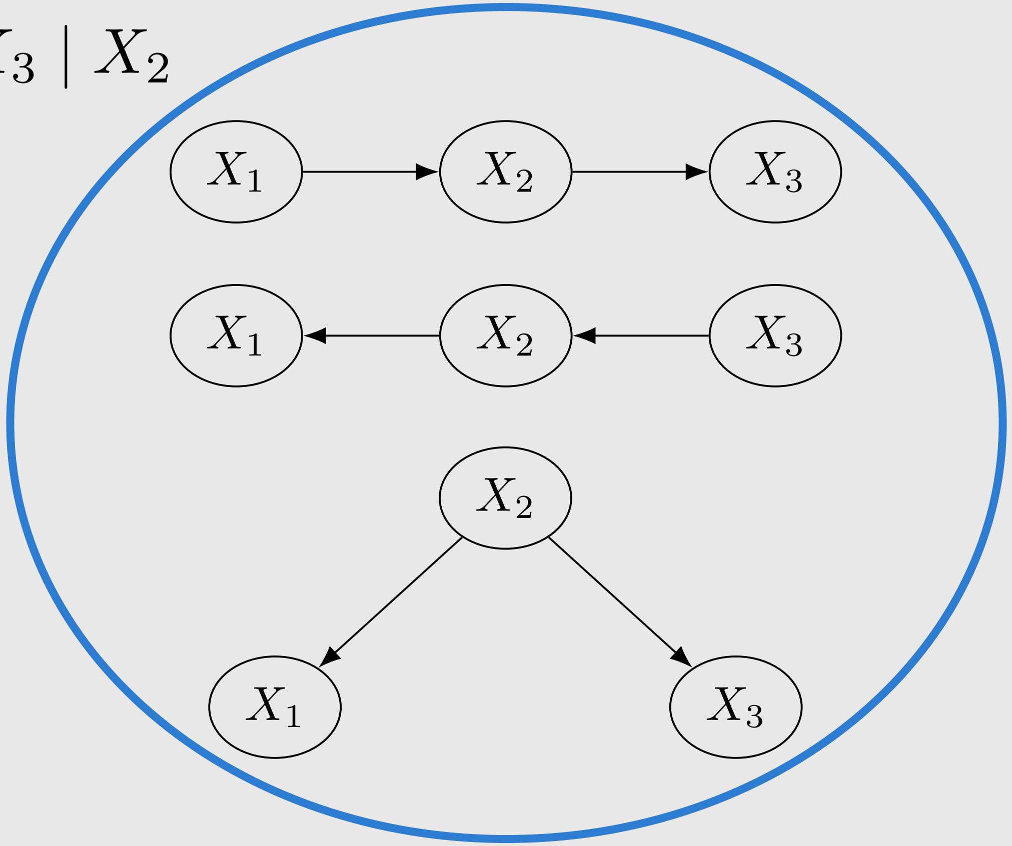


Skeletons

$X_1 \perp\!\!\!\perp X_3 \mid X_2$



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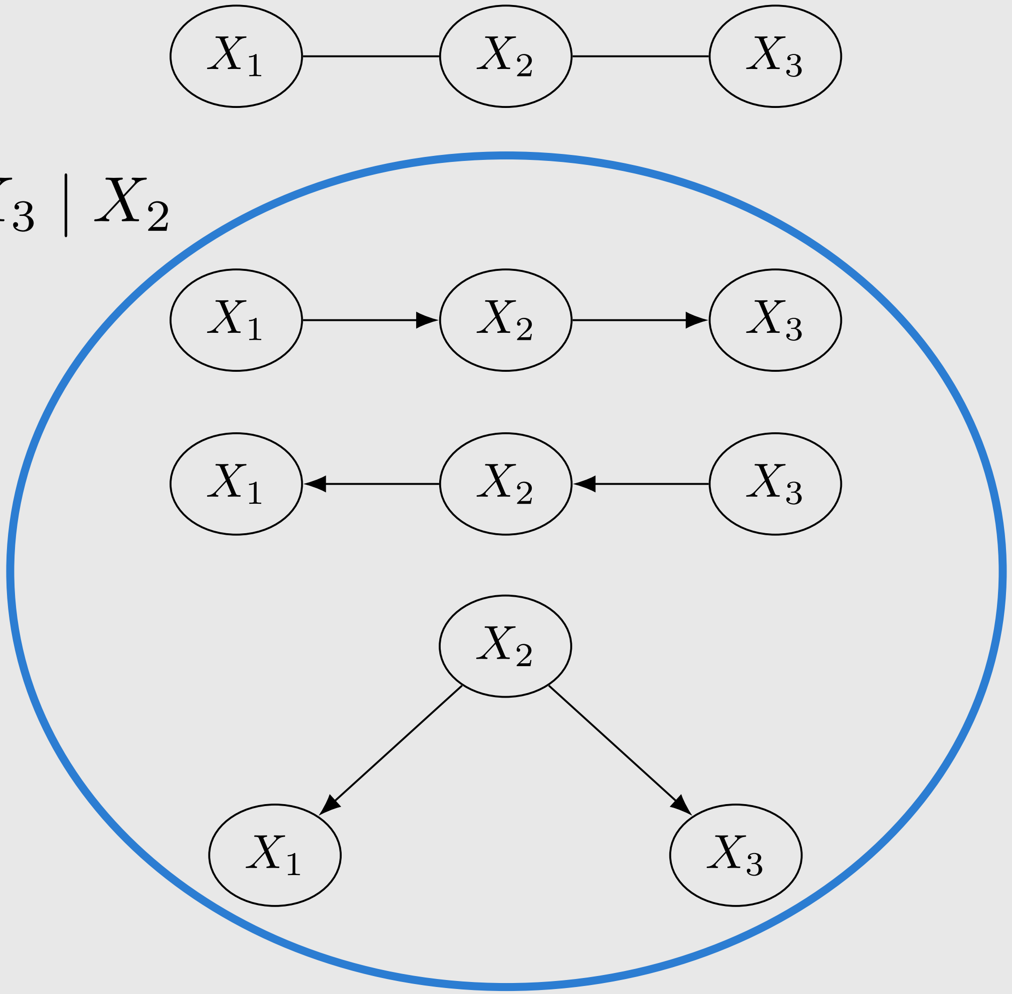

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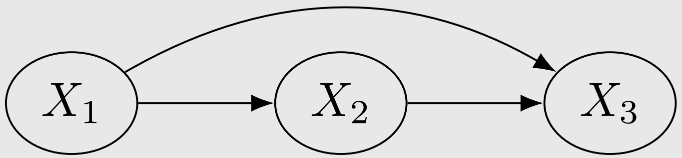
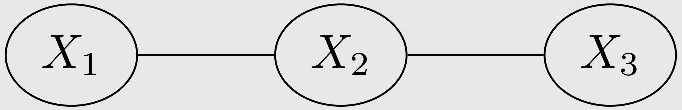
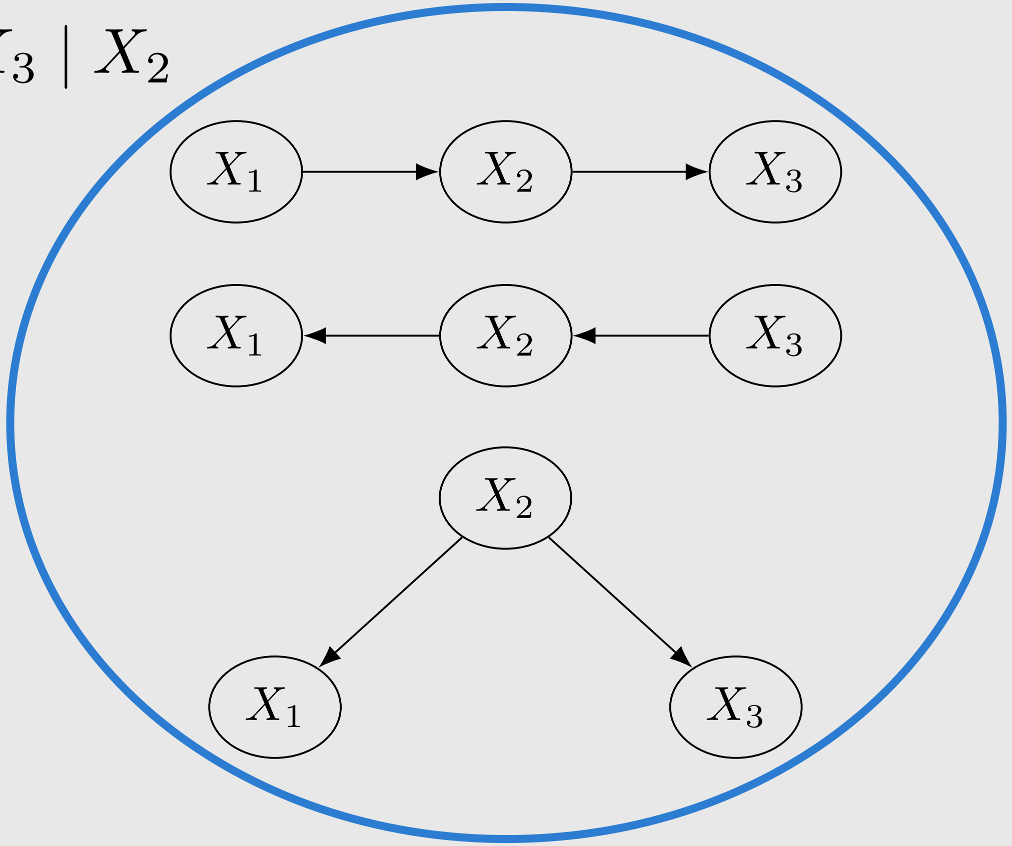




$X_1 \perp\!\!\!\perp X_3 \mid X_2$



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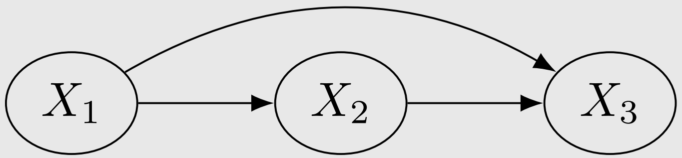

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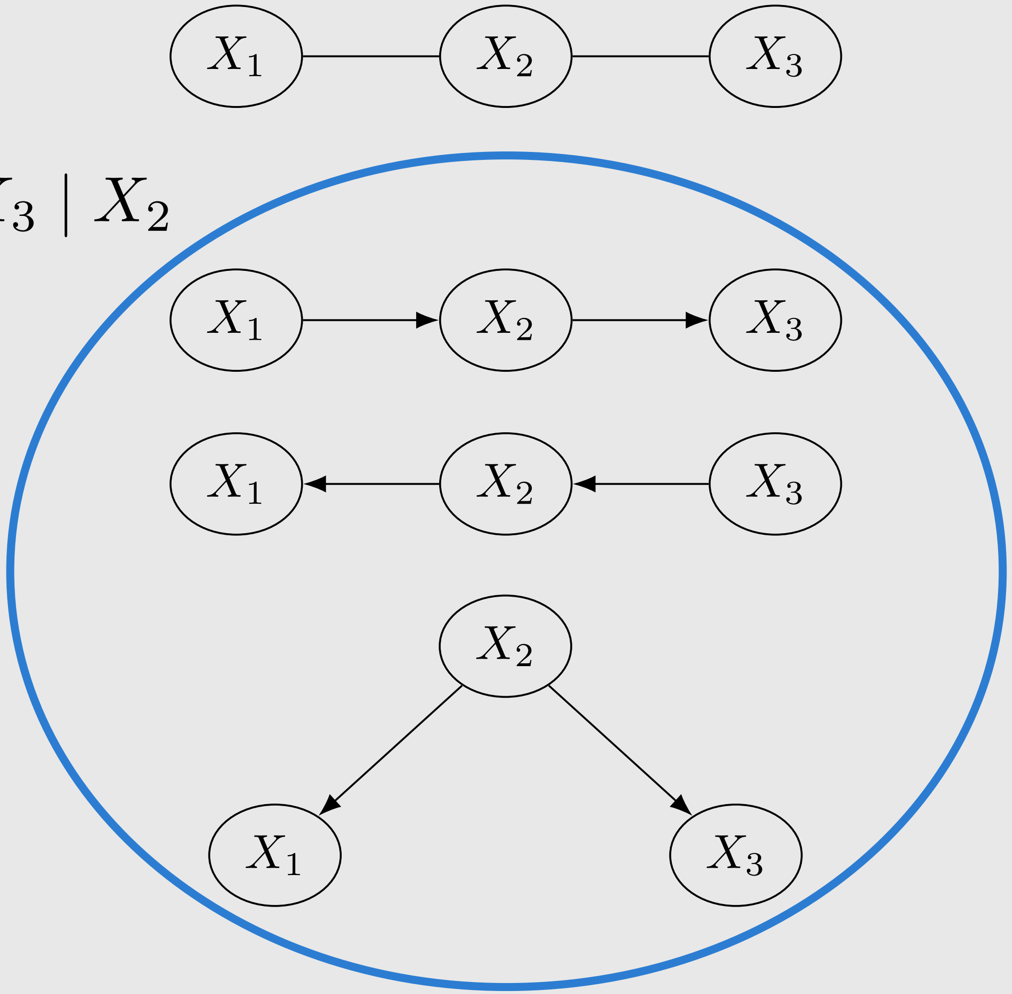

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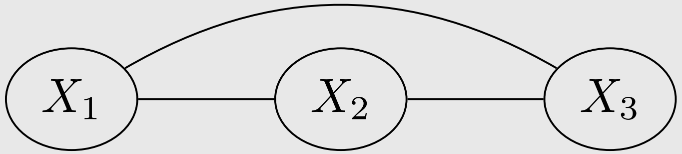

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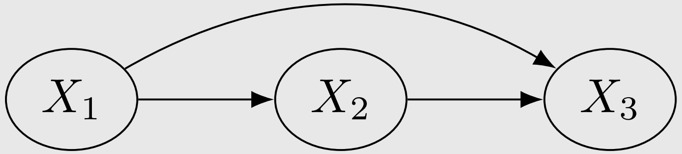

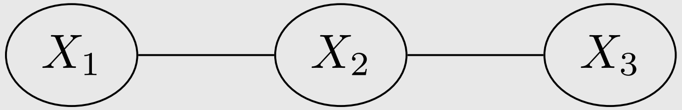

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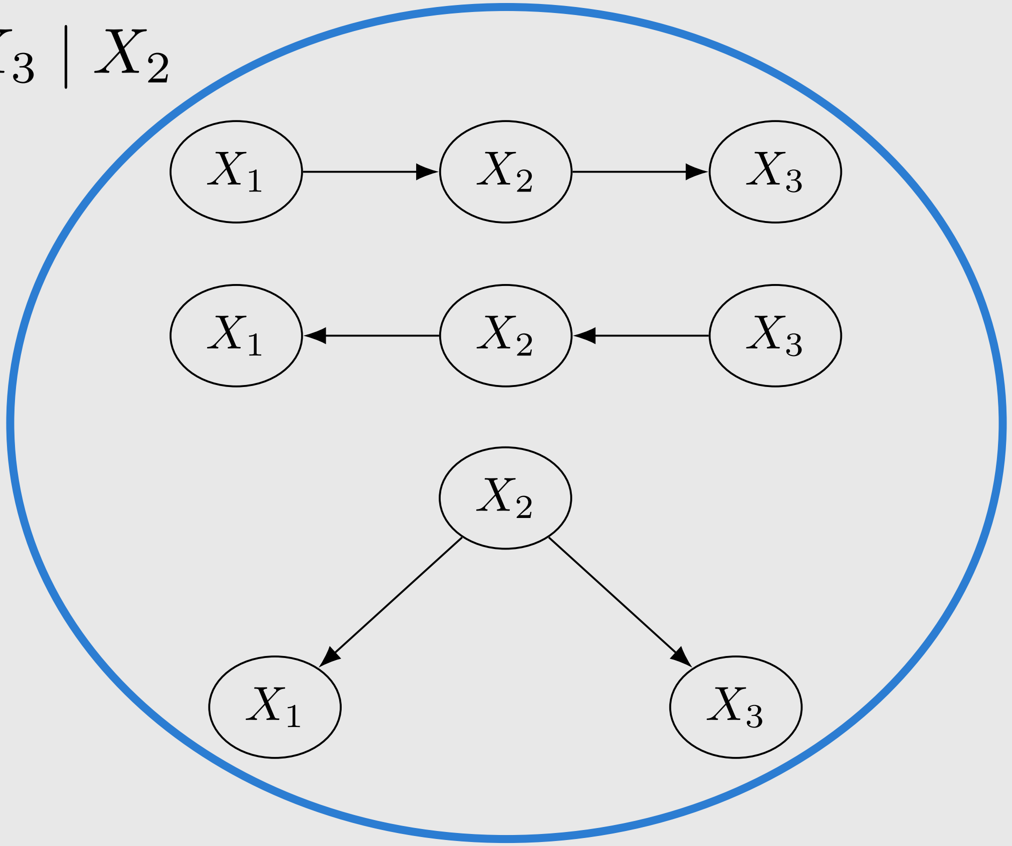

Markov Equivalence via Immoral Skeletons

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Two important graph qualities that we can use to distinguish graphs:

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Markov Equivalence via Immoral Skeletons

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Theorem: Two graphs are Markov equivalent if and only if they have the same skeleton and same immoralities (Verma & Pearl, 1990; Frydenburg, 1990).

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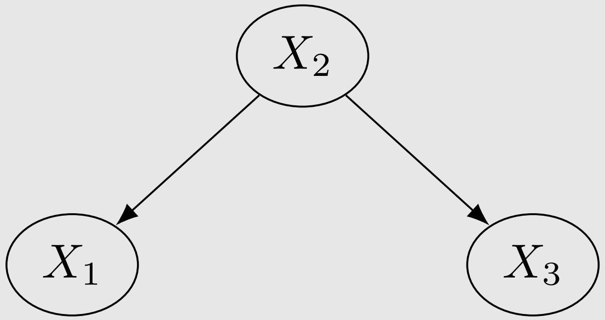
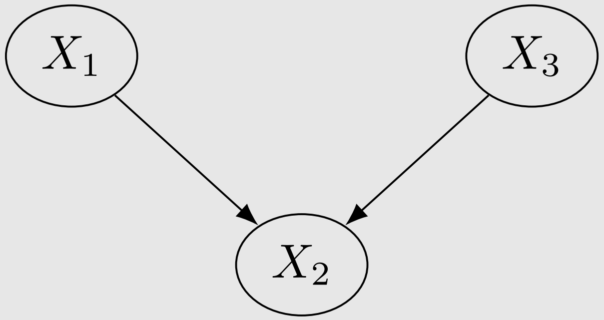
Theorem: Two graphs are Markov equivalent if and only if they have the same skeleton and same immoralities (Verma & Pearl, 1990; Frydenburg, 1990).

Essential graph (aka CPDAG): skeleton + immoralities

Question:
What graphs are Markov equivalent to the basic fork graph?



Question:
What graphs are Markov equivalent to the basic immorality?



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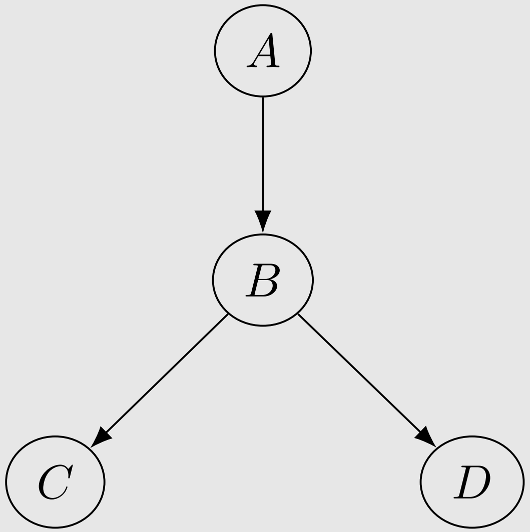

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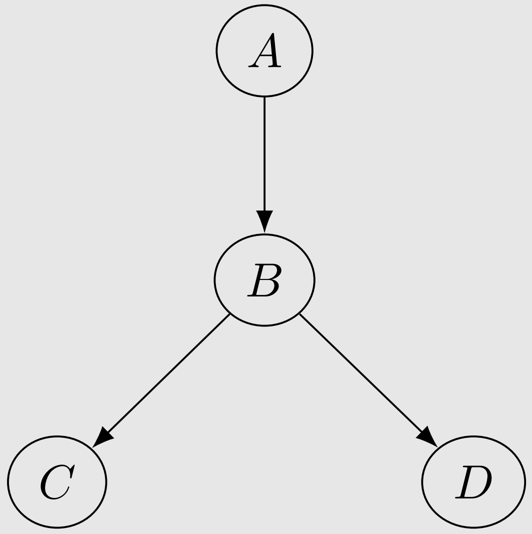

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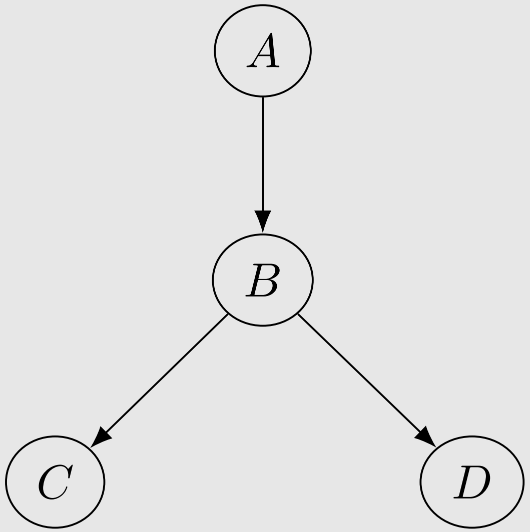

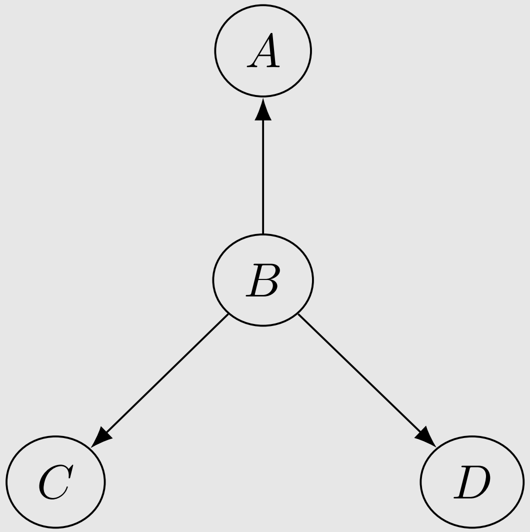

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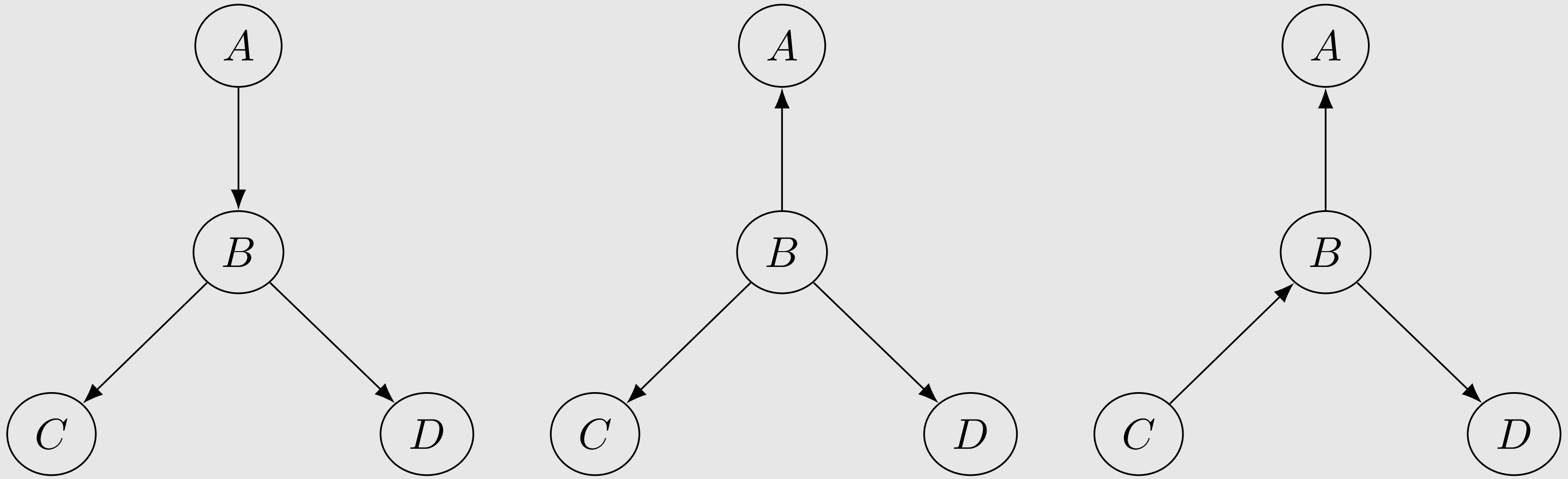

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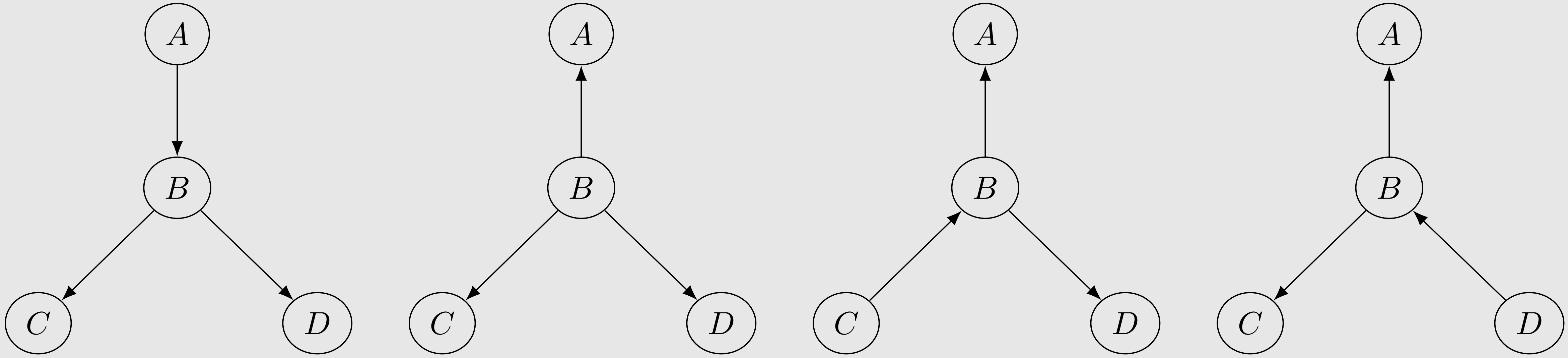

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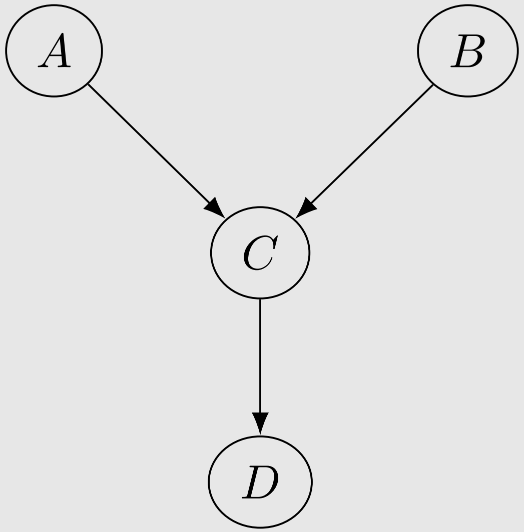
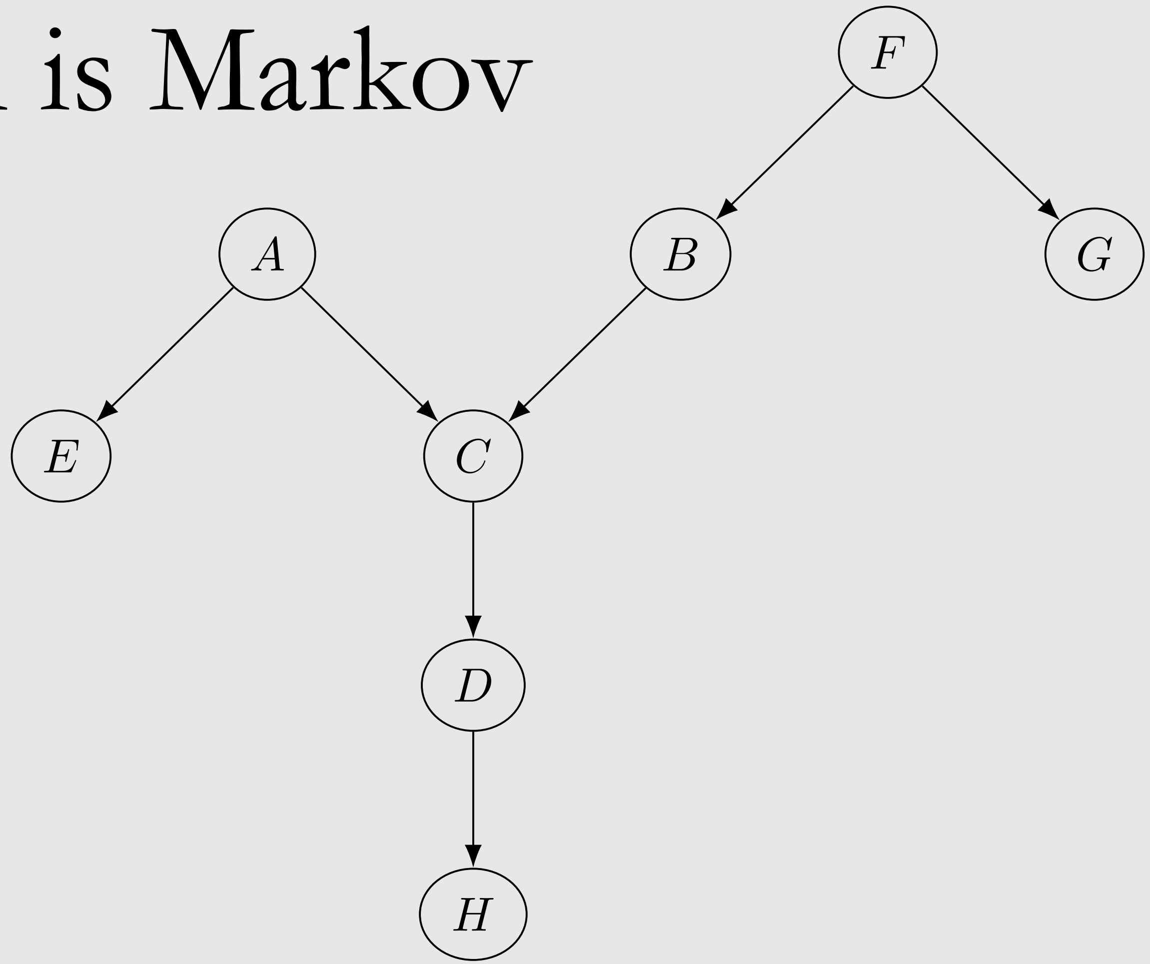


Question:
Give a few graphs that the following graph is Markov equivalent to:



The PC Algorithm: Overview

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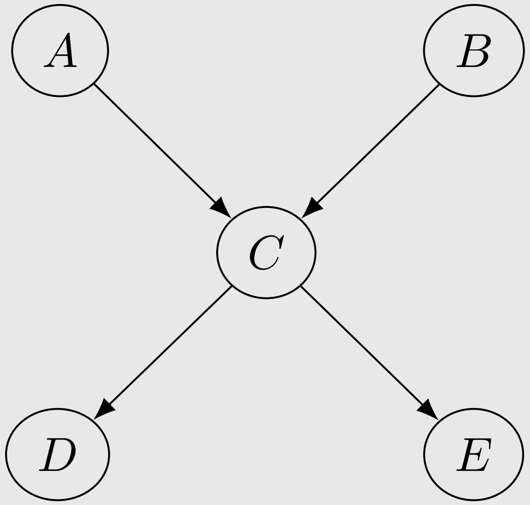
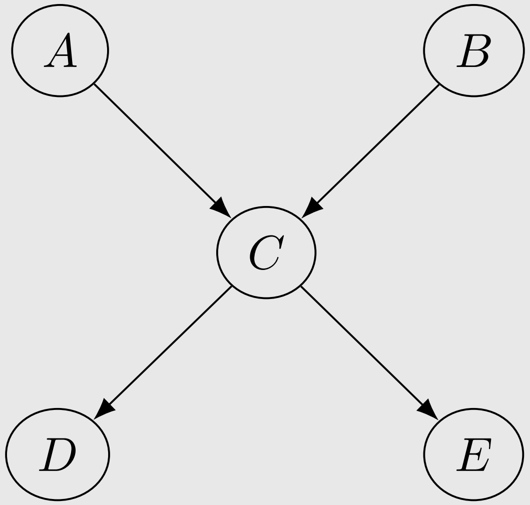
True graph



The PC Algorithm: Overview

Start with complete undirected graph

True graph



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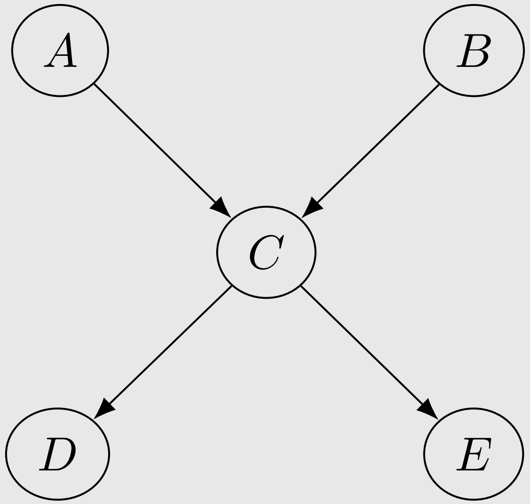
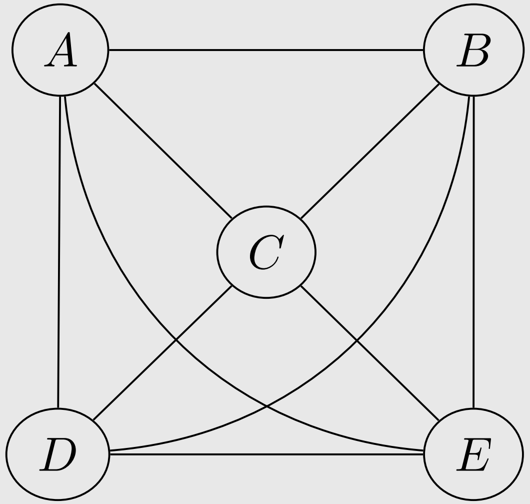


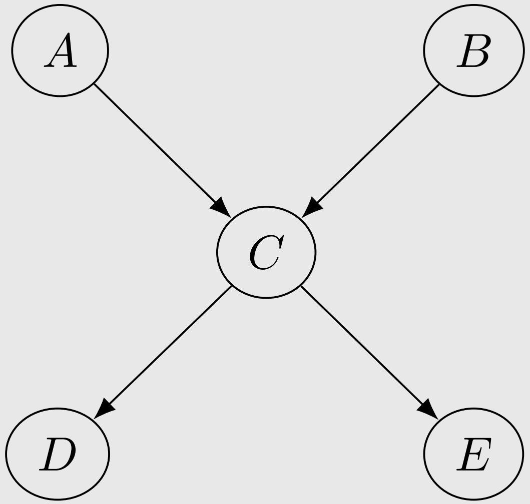


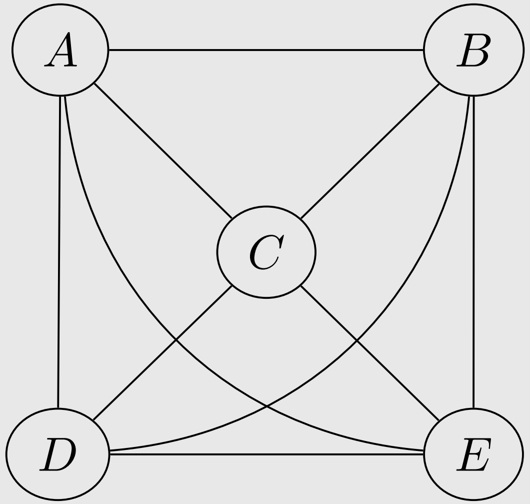
The PC Algorithm: Overview

Start with complete undirected graph

Three steps:

True graph





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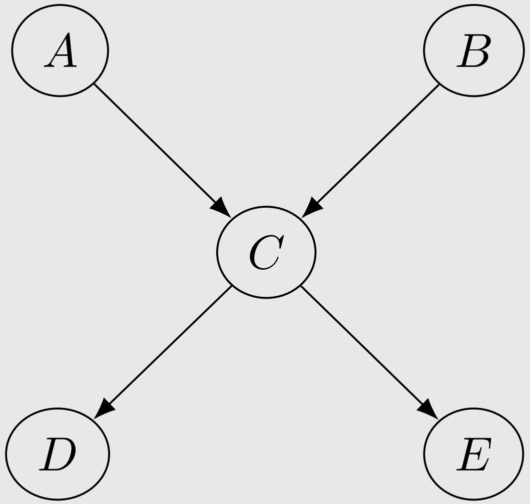
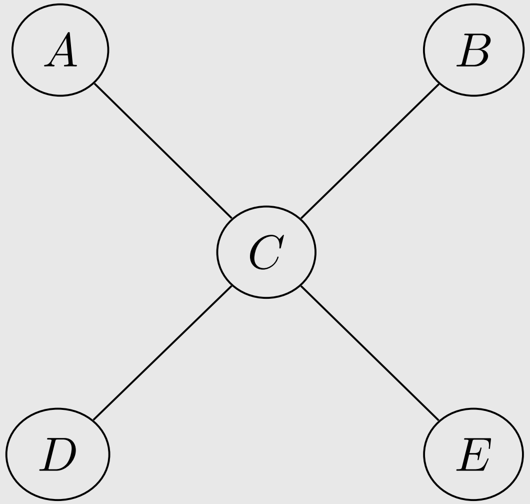


Start with complete undirected graph

Three steps:

1. Identify the skeleton



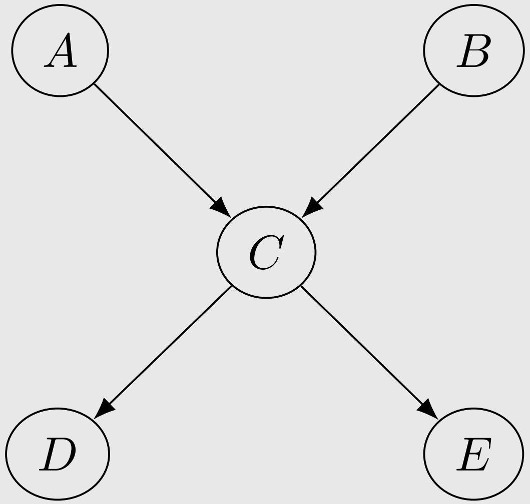
The PC Algorithm: Overview

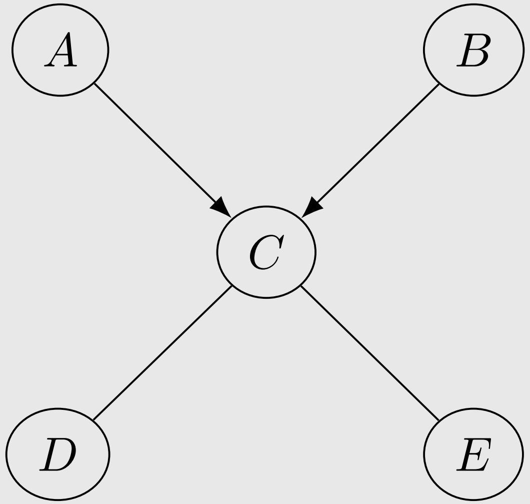
Start with complete undirected graph

Three steps:

1. Identify the skeleton
2. Identify immoralities and orient them

True graph





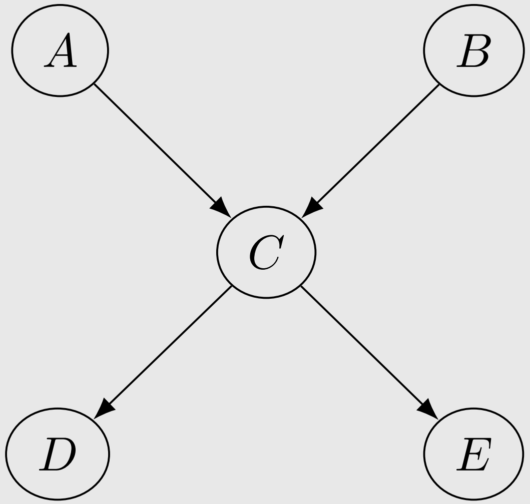
The PC Algorithm: Overview

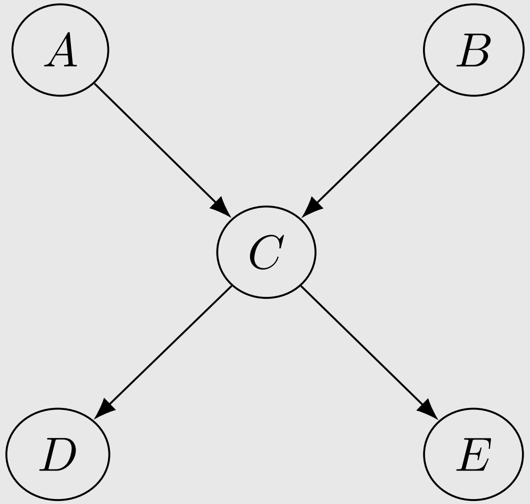
Start with complete undirected graph

Three steps:

1. Identify the skeleton
2. Identify immoralities and orient them
3. Orient qualifying edges that are incident on colliders

True graph





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Start with complete undirected graph and remove edges X – Y where $X \perp\!\!\!\perp Y \mid Z$ for some (potentially empty) conditioning set Z, starting with the empty conditioning set and increasing the size

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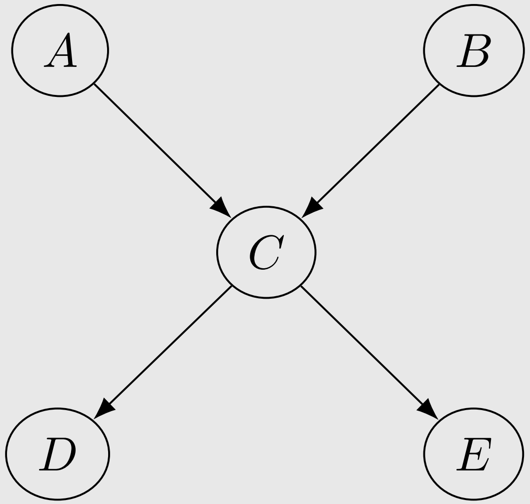

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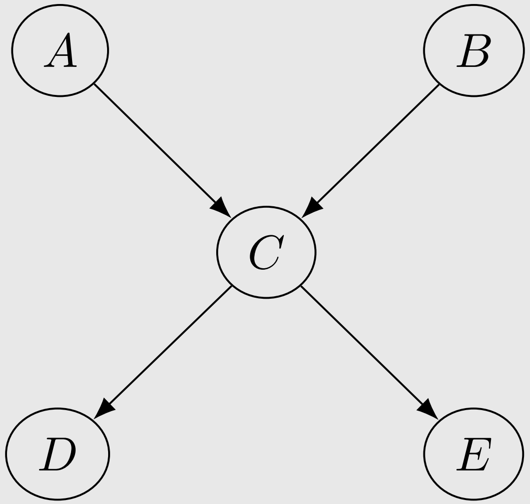

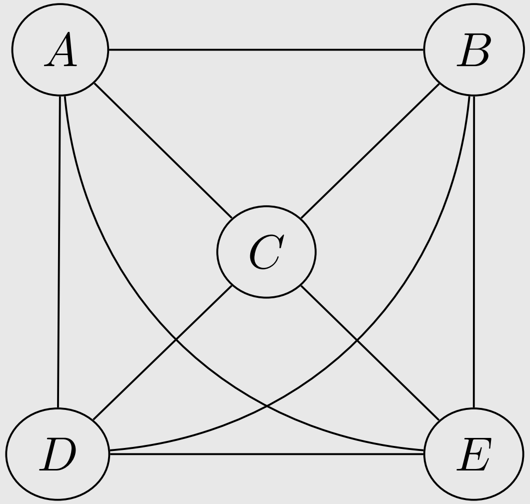

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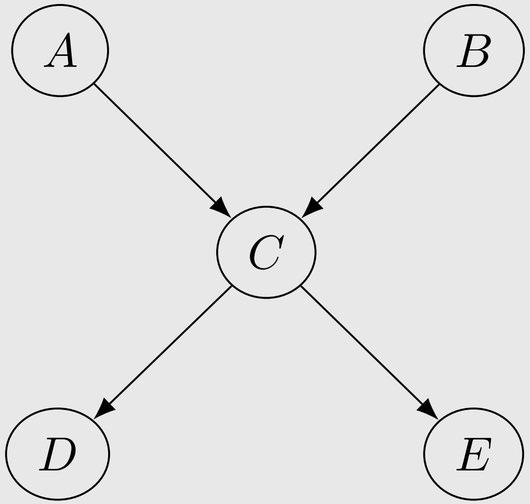

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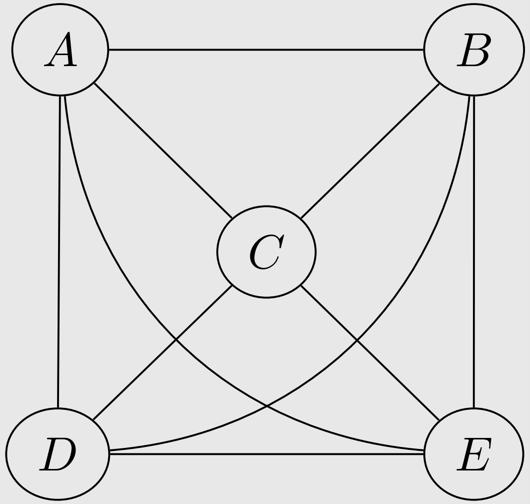

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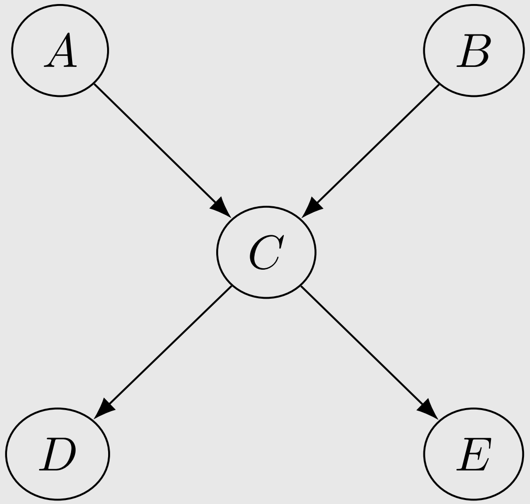

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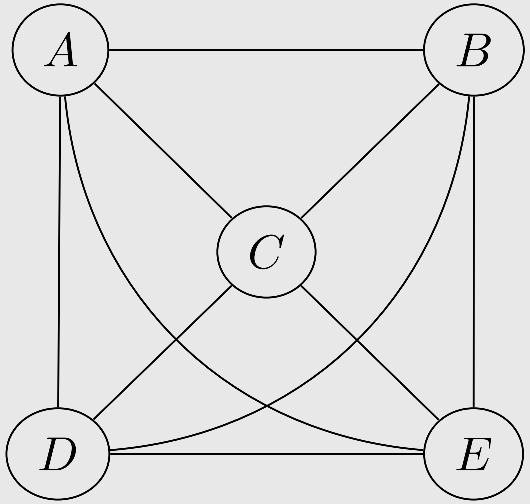

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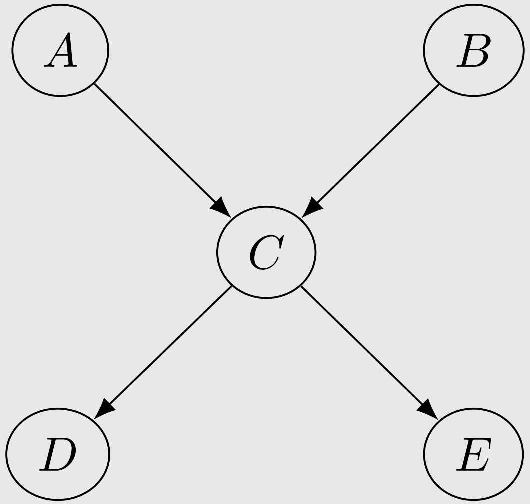

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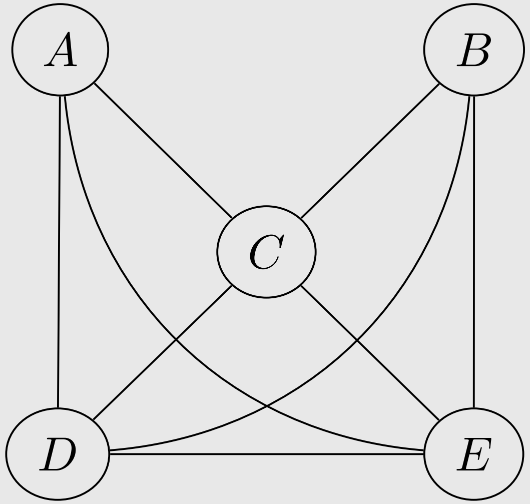

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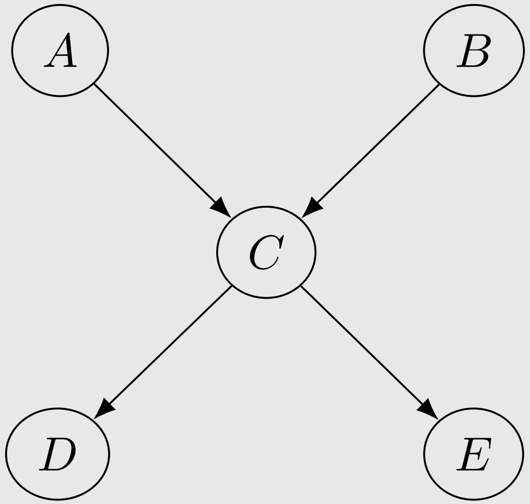

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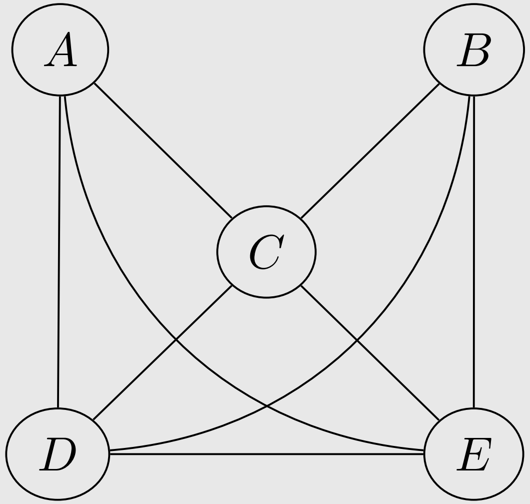

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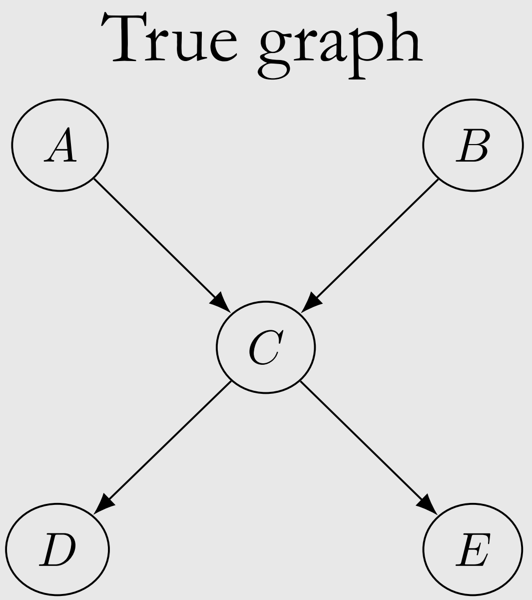

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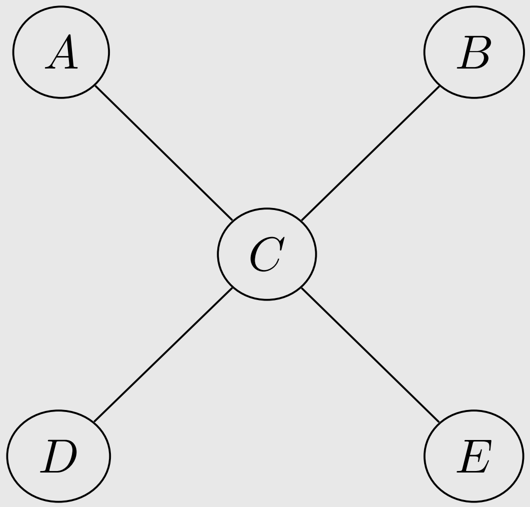
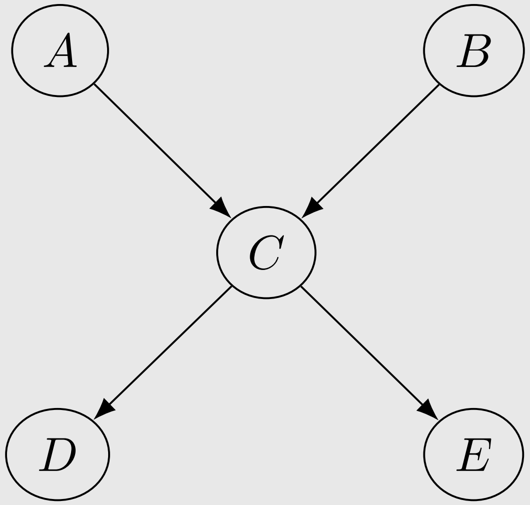
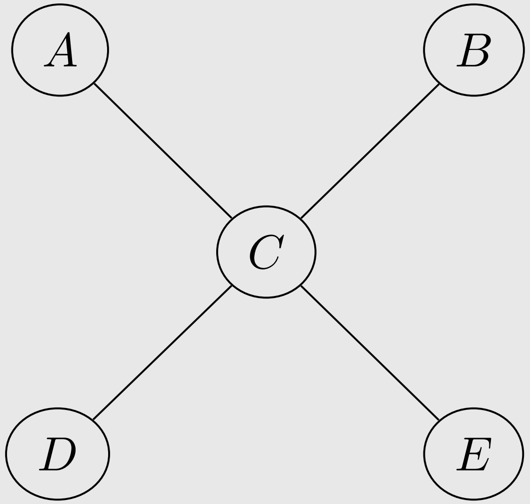


Identifying the Immoralities

True graph





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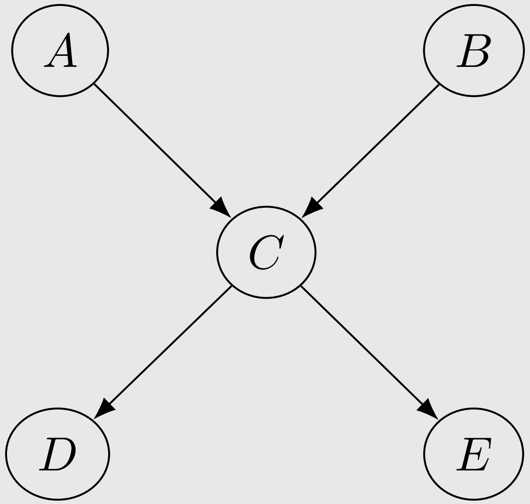

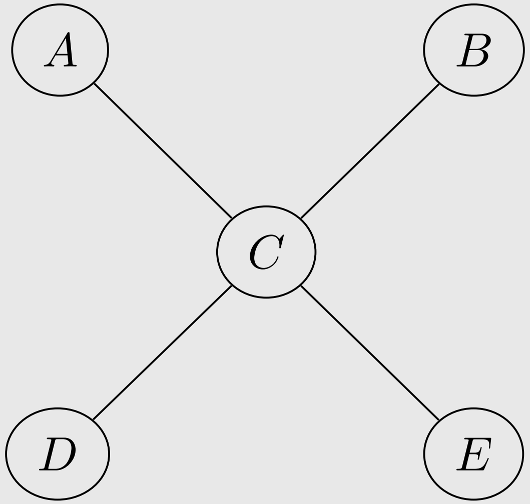

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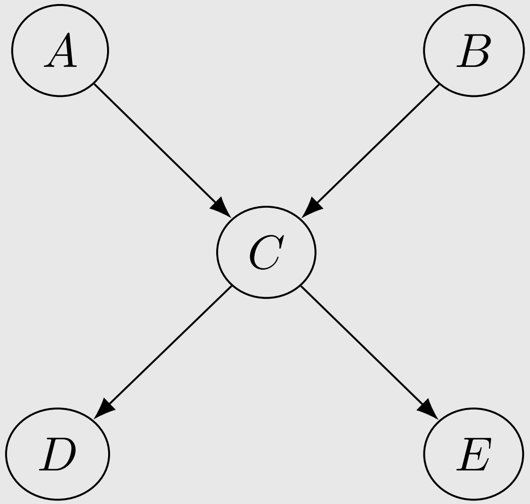

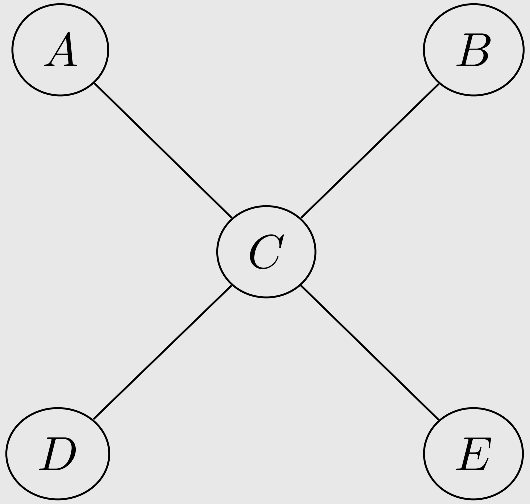

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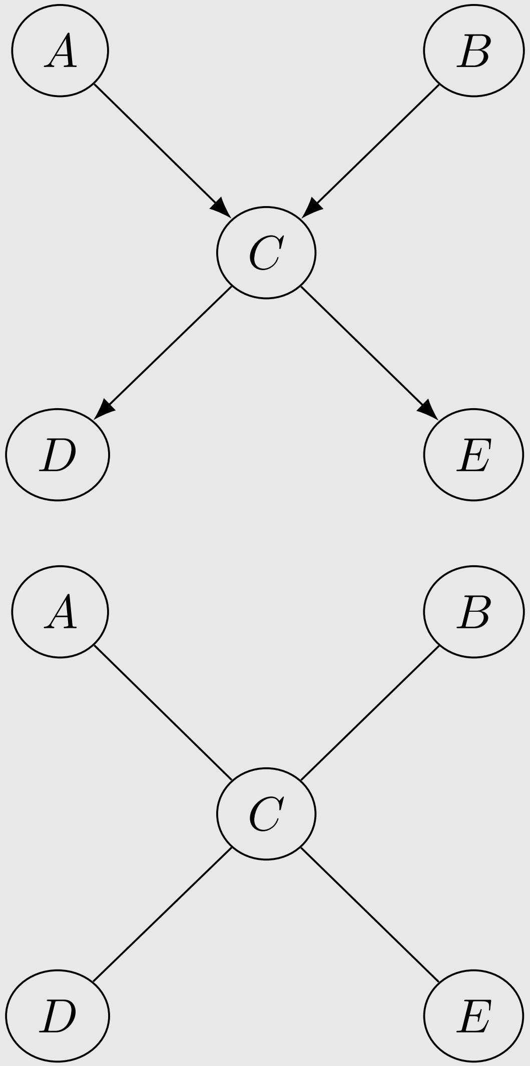

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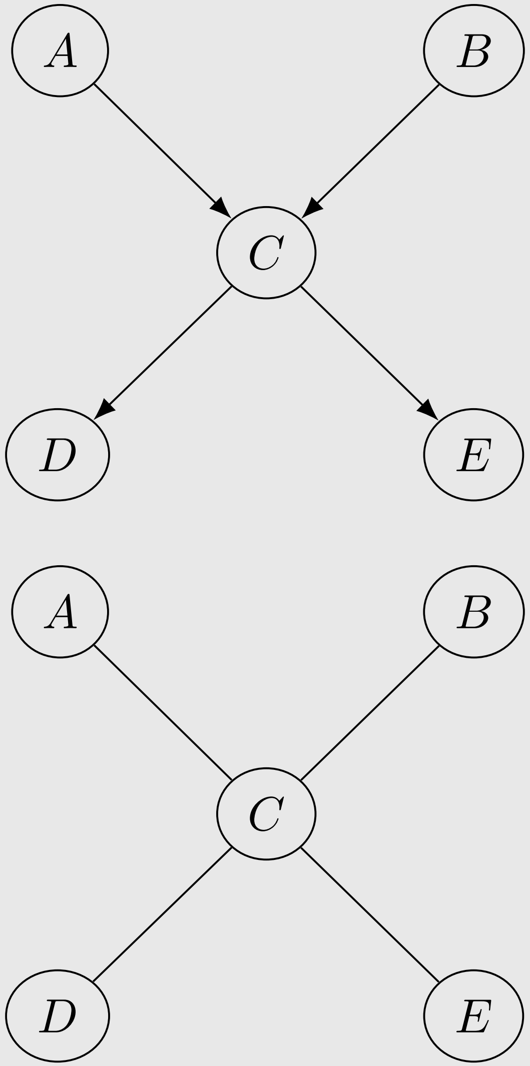

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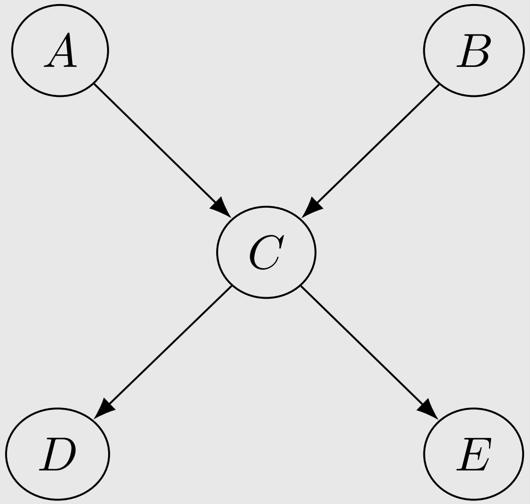

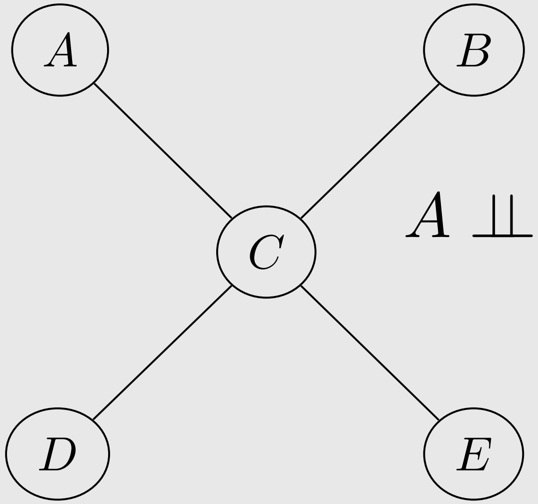

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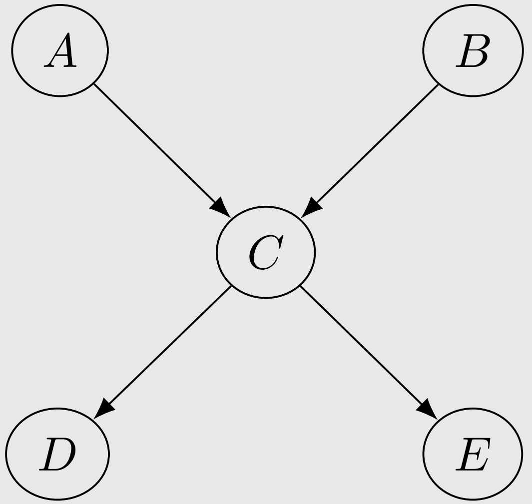

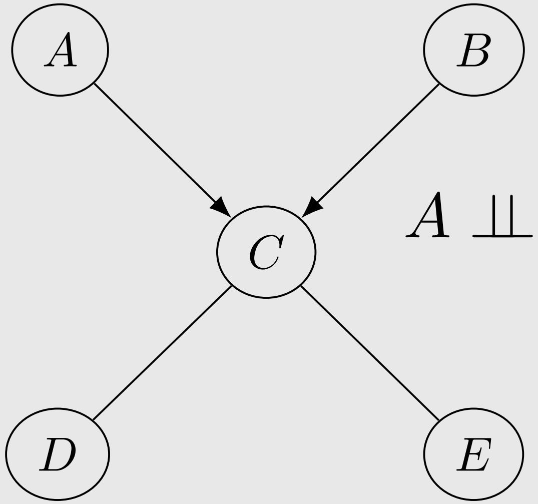

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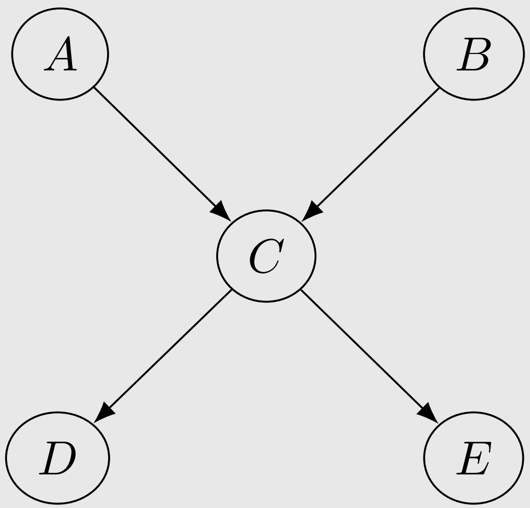
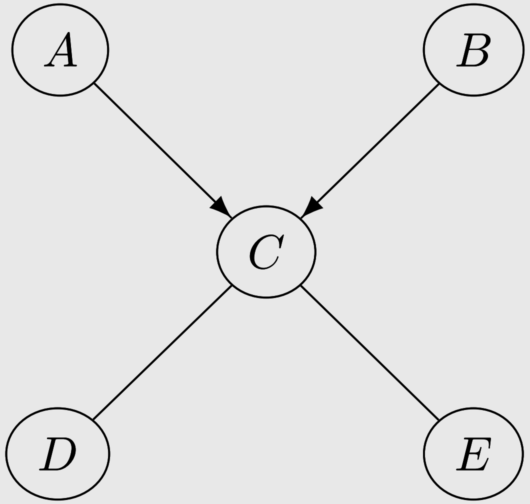
Orienting Qualifying Edges Incident on Colliders

True graph

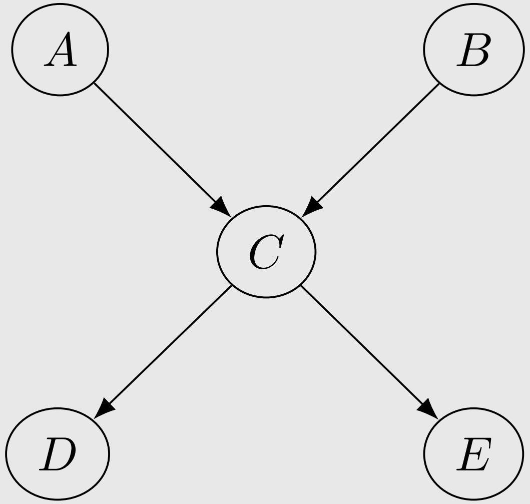


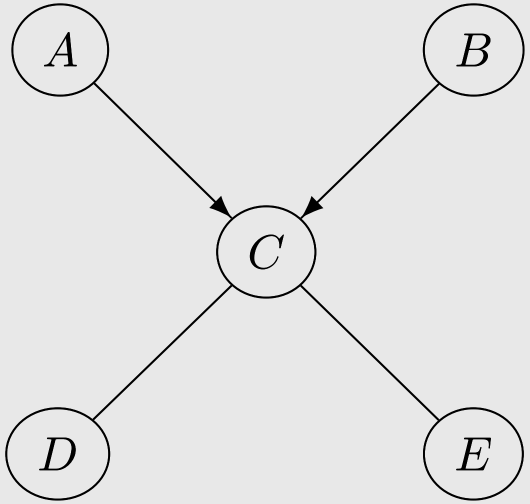


Orienting Qualifying Edges Incident on Colliders

Idea: use fact that we discovered all immoralities to orient more edges

True graph





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Any edge $Z - Y$ part of a partially directed path of the form $X \to Z - Y$, where there is no edge connecting X and Y can be oriented as $Z \to Y$

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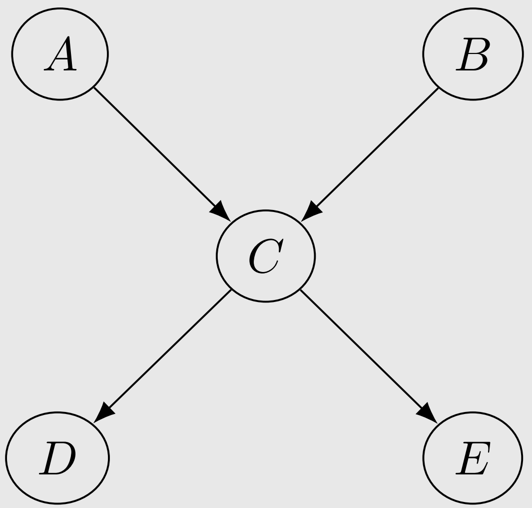

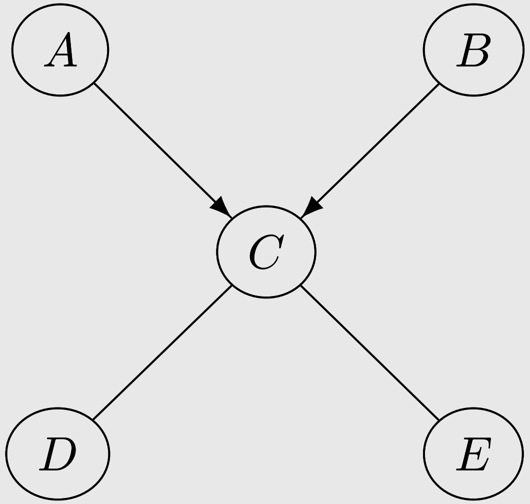

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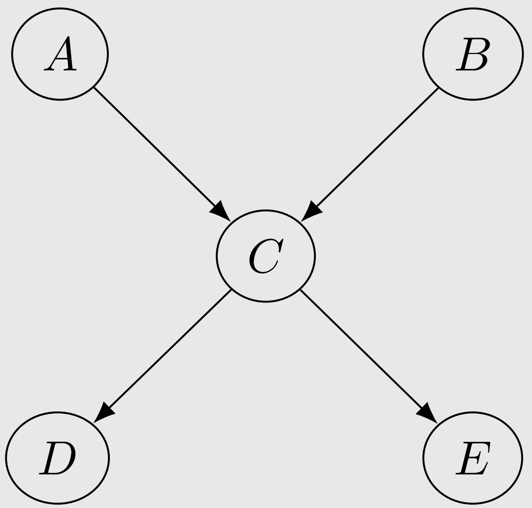

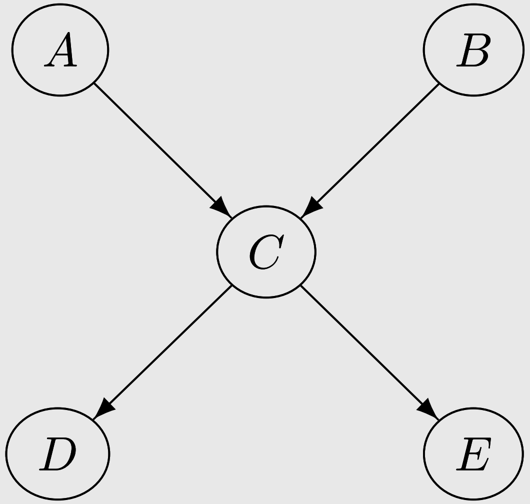

Removing Assumptions

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Neither causal sufficiency nor acyclicity: SAT-based causal discovery (Hyttinen et al., 2013; 2014)

Hardness of Conditional Independence Testing

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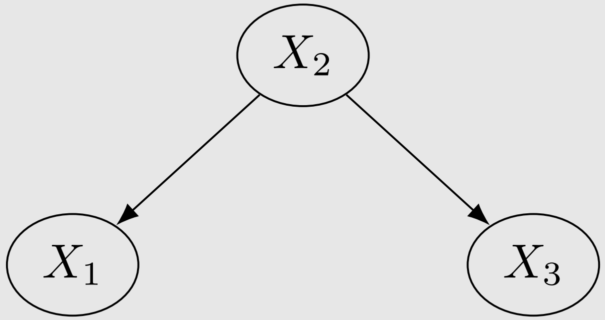
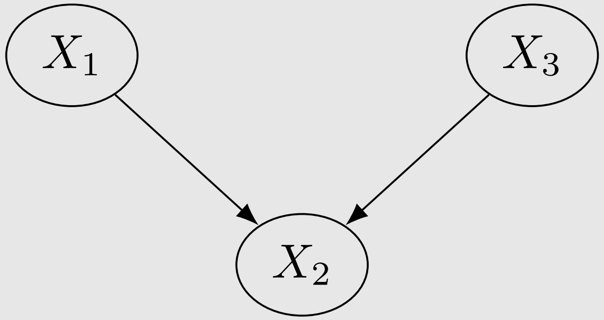
Conditional independence testing is simple if we have infinite data.

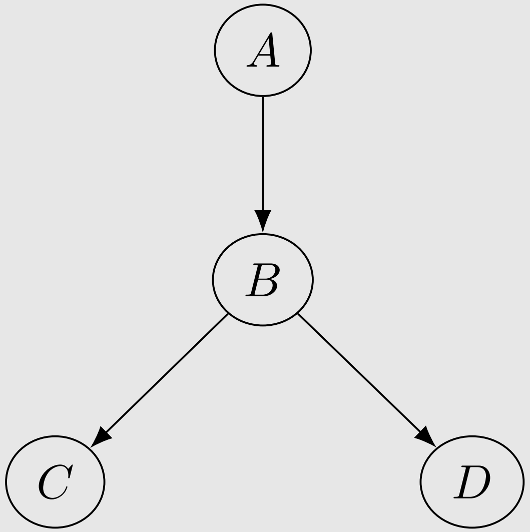
However, it is a quite hard problem with finite data, and it can sometimes require a lot of data to get accurate test results (Shah & Peters, 2020).

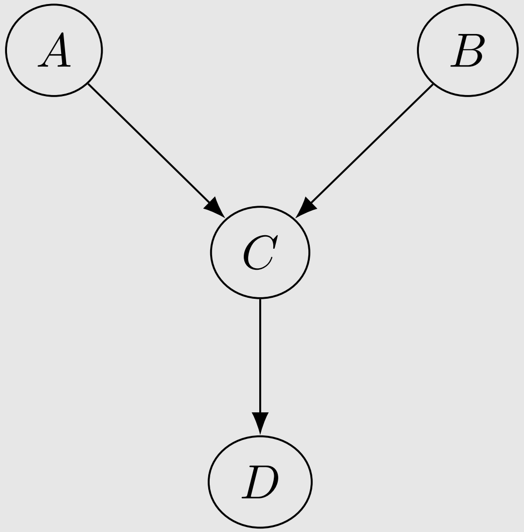
Questions:

1. What are the essential graphs of the following graphs?









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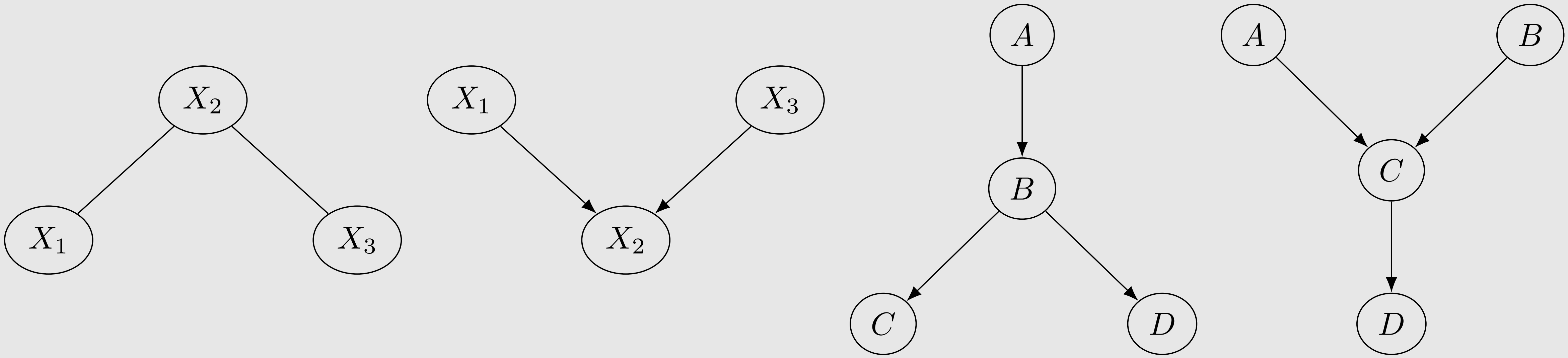

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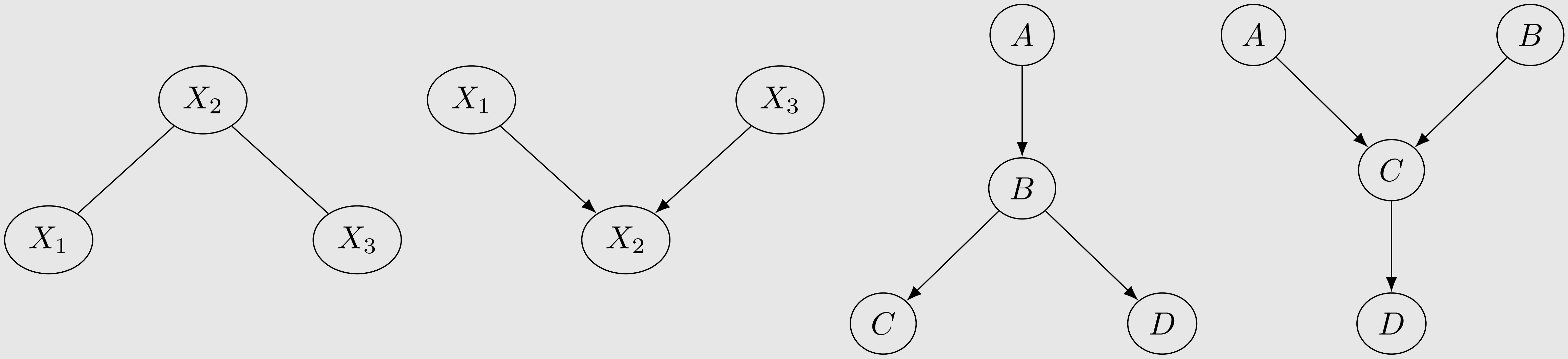

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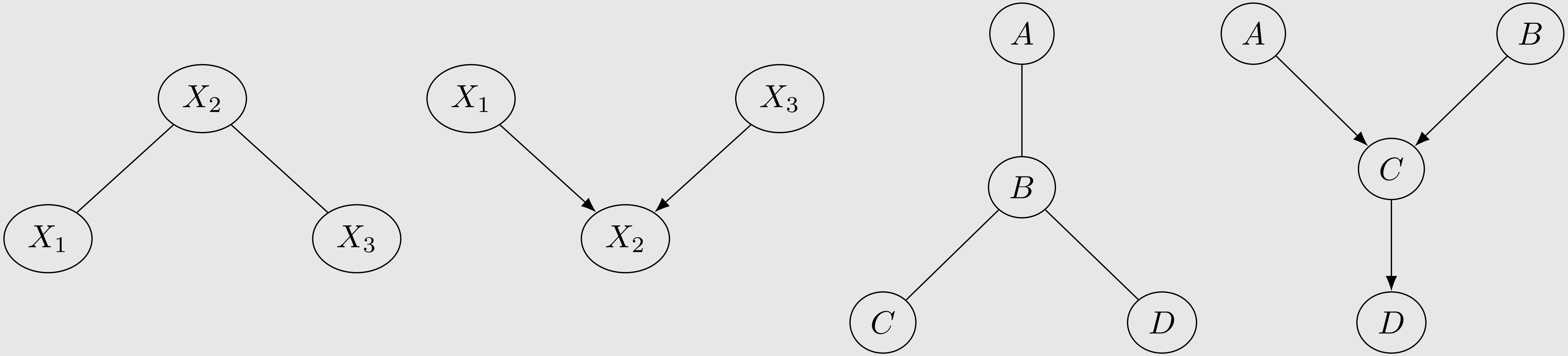

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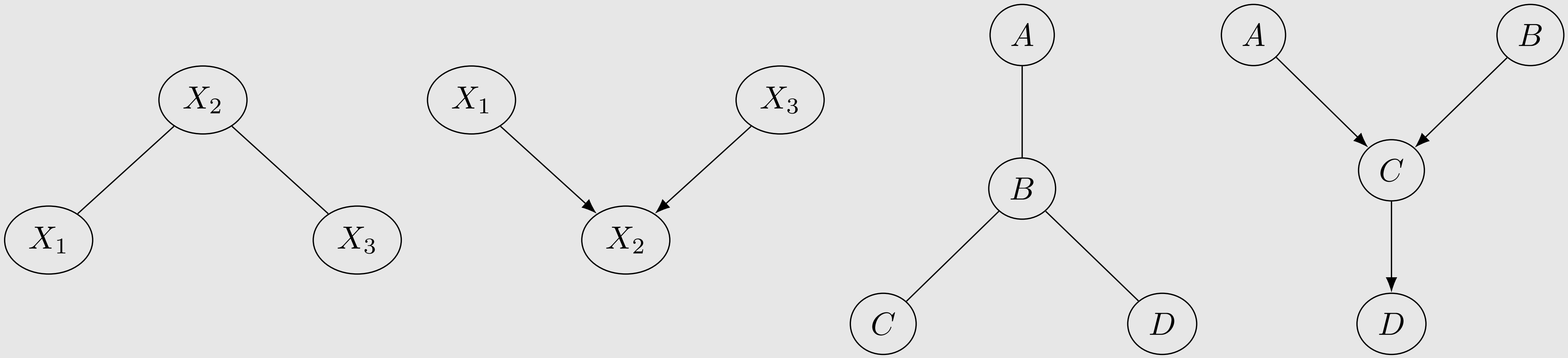
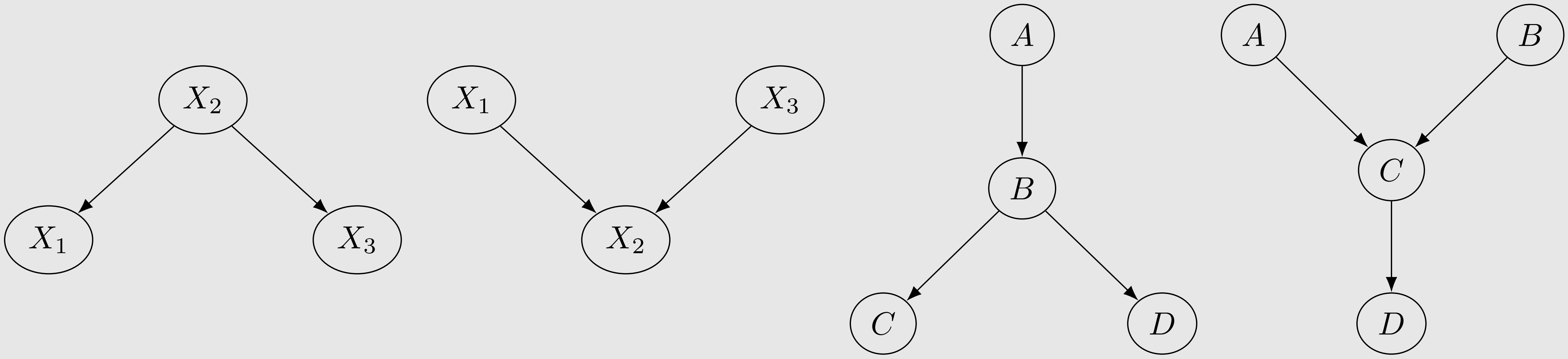
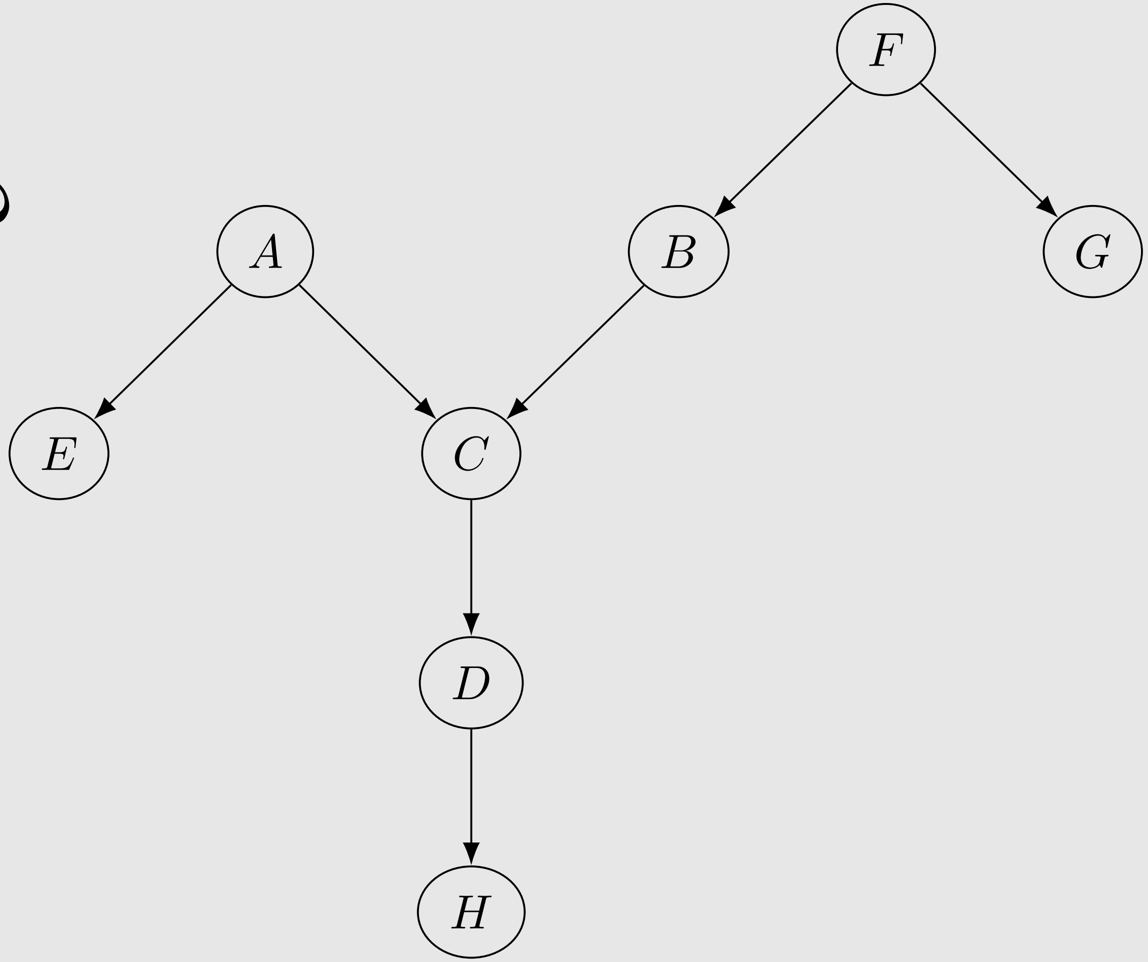


Questions:

1. What are the essential graphs of the following graphs?
2. Walk through the steps of PC to get them.



Question:
What is the essential graph for this graph?



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What about non-Gaussian structural equations?

Or nonlinear structural equations?

Independence-Based Causal Discovery

Semi-Parametric Causal Discovery

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Two Variable Case: Markov Equivalence

Infinite data: $P(x, y)$

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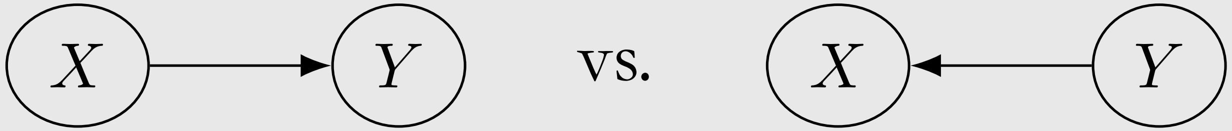

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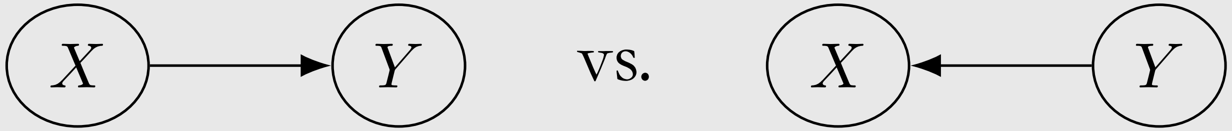
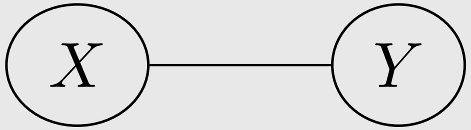


Essential graph:



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$$Y = f_Y(X, U_Y), \quad X \perp\!\!\!\perp U_Y$$

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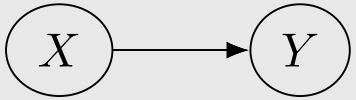

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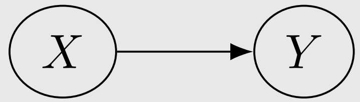

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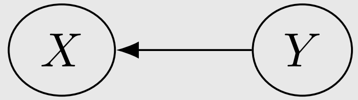

where U_Y and U_X are real-valued random variables.

We must make assumptions about the parametric form.

Linear Non-Gaussian Assumption

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Linear Non-Gaussian Assumption:

All structural equations (causal mechanisms that generate the data) are of the following form:

$$Y := f(X) + U$$

where f is a linear function, $X \perp\!\!\!\perp U$, and U is distributed as some non-Gaussian.

Identifiability in Linear Non-Gaussian Setting

Theorem (Shimizu et al., 2006):

In the linear non-Gaussian setting, if the true SCM is

$$Y := f(X) + U\,, \quad X \perp\!\!\!\perp U\,,$$

then there does not exist an SCM in the reverse direction,

$$X := g(Y) + \tilde{U}\,, \quad Y \perp\!\!\!\perp \tilde{U}\,,$$

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See proof in the course book

Identifiability in Linear Non-Gaussian Setting: Linear Fit

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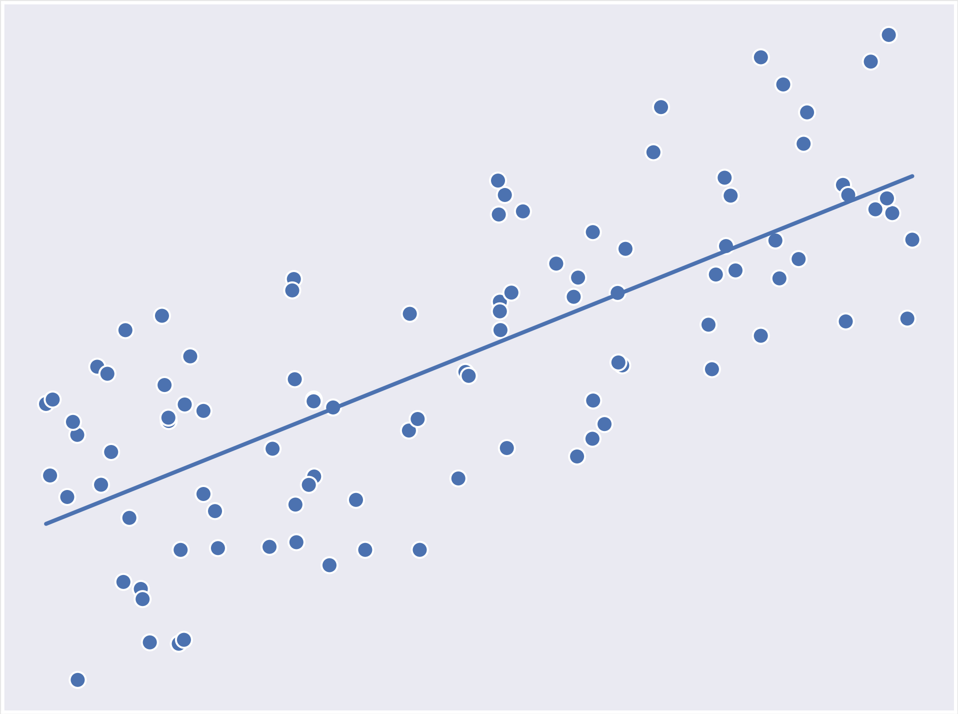
Identifiability in Linear Non-Gaussian Setting: Linear Fit



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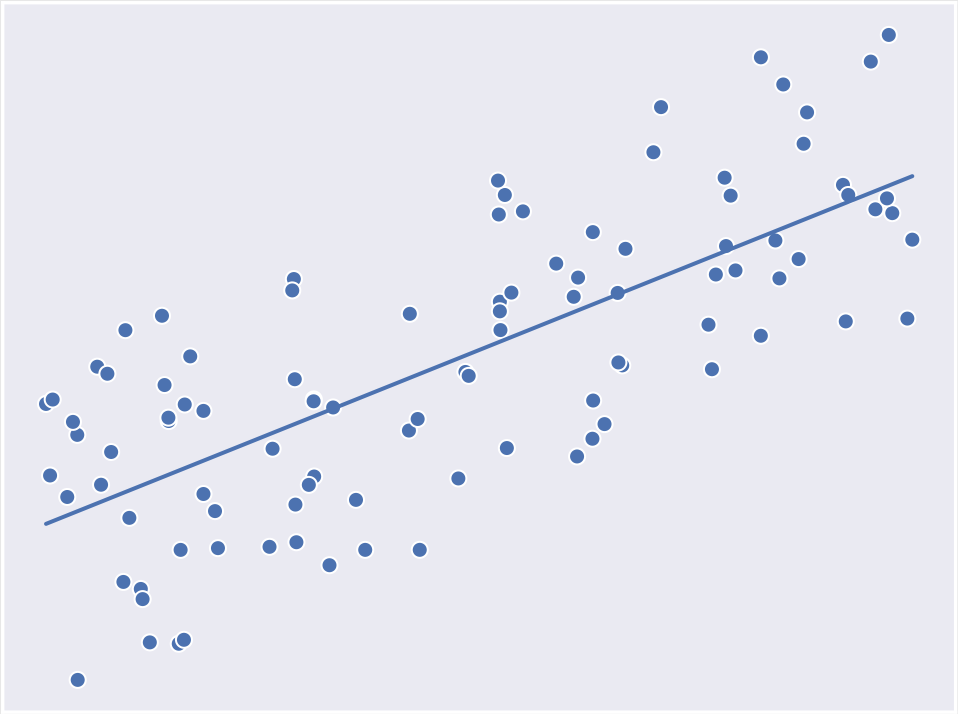
Identifiability in Linear Non-Gaussian Setting: Linear Fit



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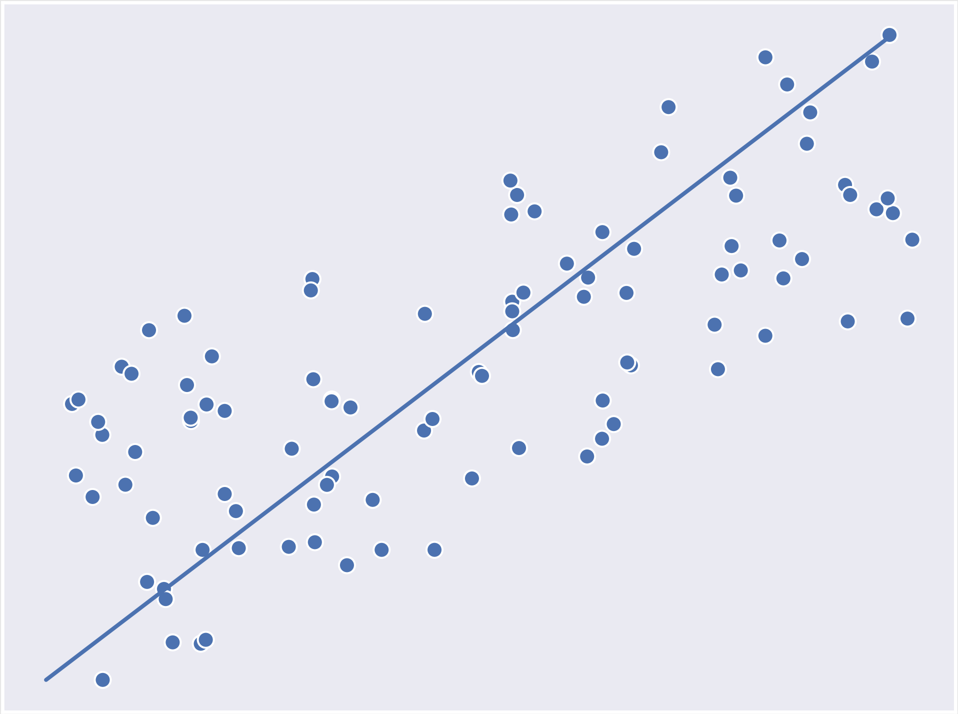

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Identifiability in Linear Non-Gaussian Setting: Residuals

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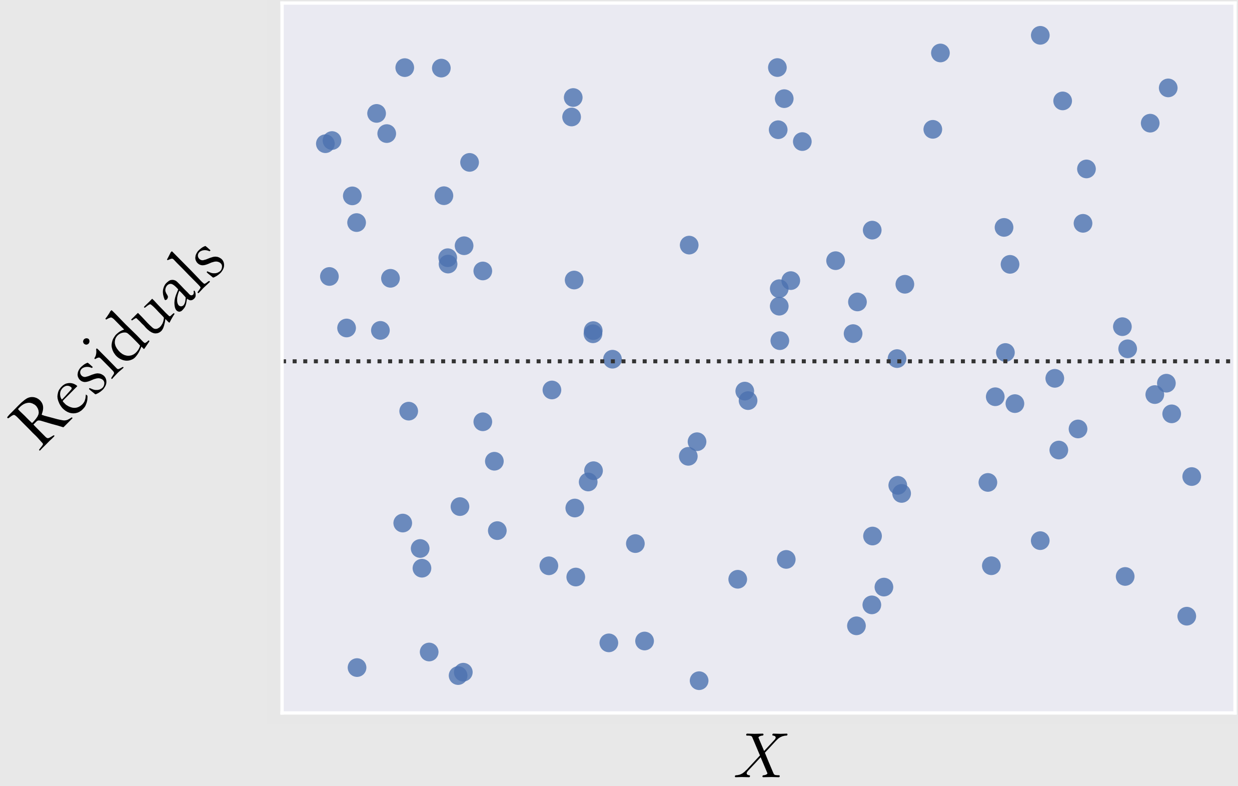
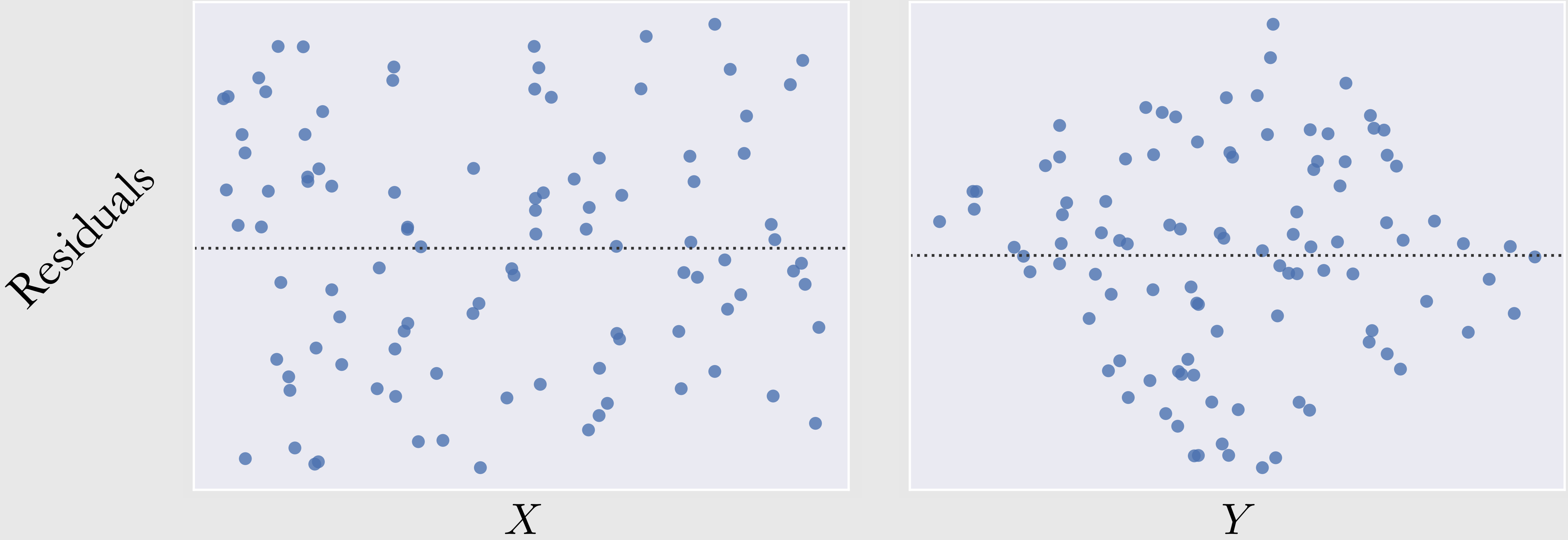


Identifiability in Linear Non-Gaussian Setting: Residuals

$$Y := f(X) + U \,, \quad X \perp\!\!\!\perp U$$

$$X := g(Y) + \tilde{U} \,, \quad Y \not\perp\!\!\!\perp \tilde{U}$$



Linear Non-Gaussian Identifiability Extensions

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- Drop causal sufficiency assumption: Hoyer et al. (2008)
- Drop acyclicity assumption: Lacerda et al. (2008)

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Theorem (Hoyer et al. 2008): Under the Markov assumption, causal sufficiency, acyclicity, the nonlinear additive noise assumption, and a technical condition from Hoyer et al. (2008), we can identify the causal graph.

Post-Nonlinear Setting

Nonlinear additive noise setting: $Y := f(X) + U, \quad X \perp\!\!\!\perp U$

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$$Y := g(f(X) + U)\,, \quad X \perp\!\!\!\perp U$$

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Nonlinear additive noise setting: $Y := f(X) + U \,, \quad X \perp\!\!\!\perp U$

Post-nonlinear (Zhang & Hyvärinen, 2009):

$$Y := g(f(X) + U) \,, \quad X \perp\!\!\!\perp U$$

Review Articles and Book

- Introduction to the Foundations of Causal Discovery (Eberhardt, 2017)
- Review of Causal Discovery Methods Based on Graphical Models (Glymour, Zhang, & Spirtes, 2019)
- Elements of Causal Inference (Peters, Janzing, & Schölkopf, 2017)