Difference-in-Differences

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causalcourse.com

Motivation and Preliminaries

Difference-in-Differences Overview

Assumptions and Proof

Problems with Difference-in-Differences

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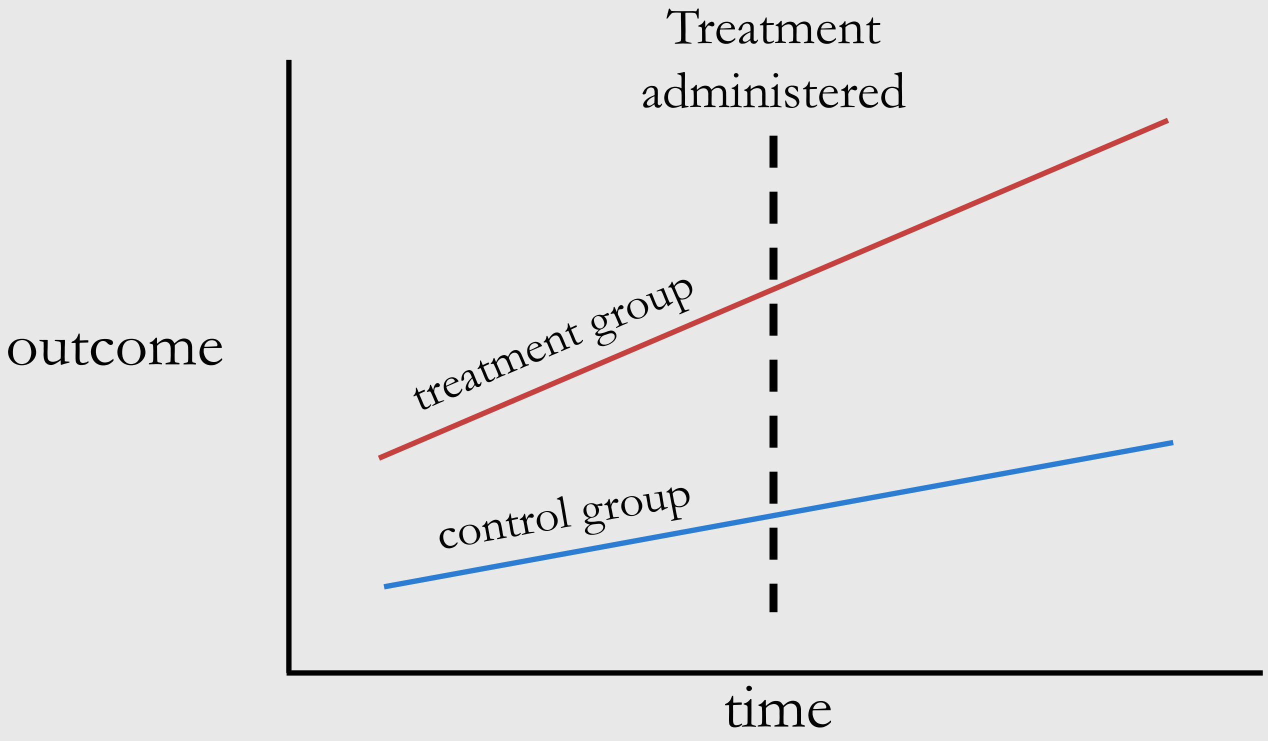
Assumptions and Proof

Problems with Difference-in-Differences

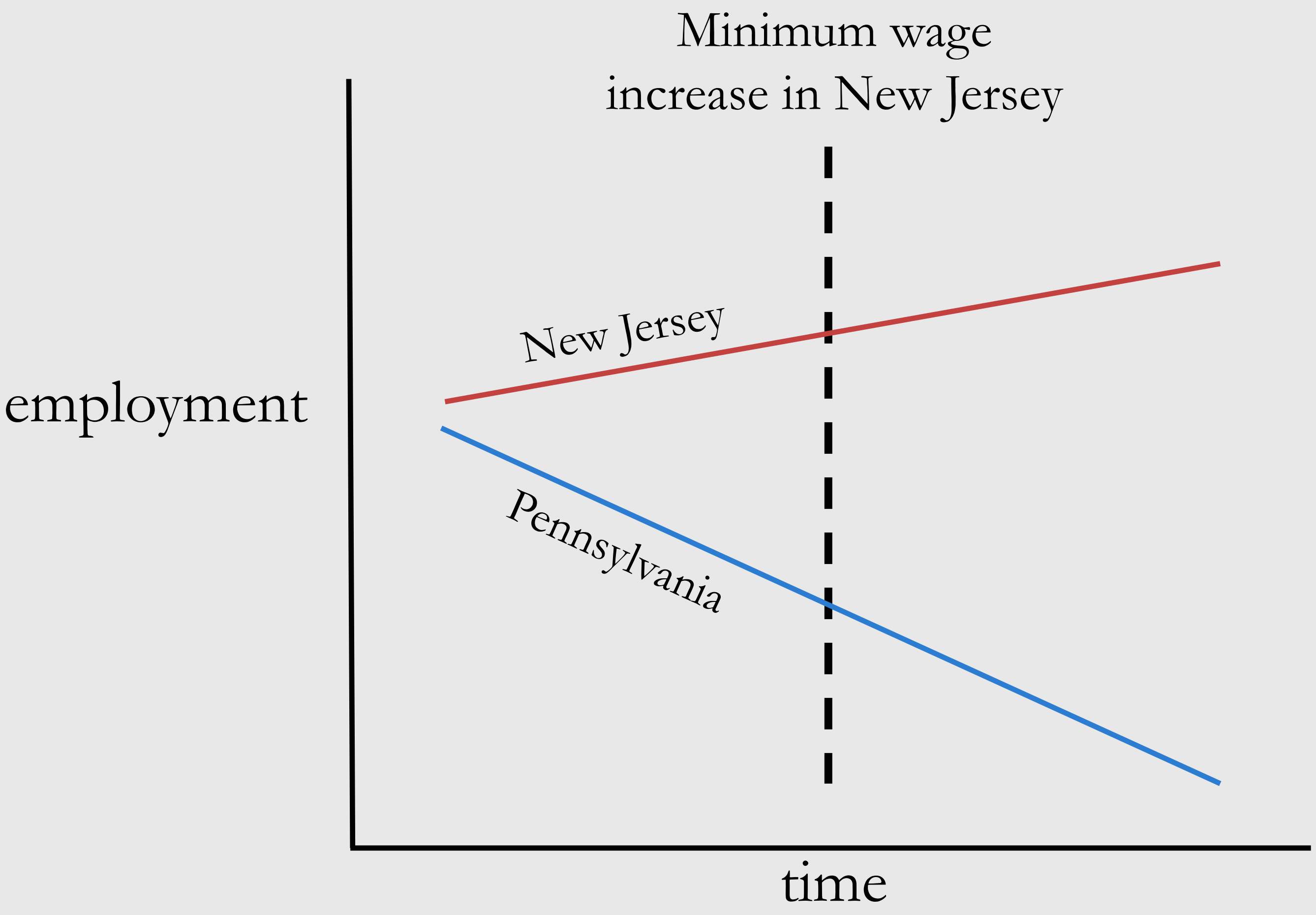
Motivation

Context: Treatment and control group both before and after treatment administered

Advantage: Use time dimension to help with identification



Example from Card & Krueger (1994)



Average Treatment Effect on the Treated (ATT)

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ATE:

$\mathbb{E}[Y(1) - Y(0)]$

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Unconfoundedness:

$(Y(0), Y(1)) \perp\!\!\!\perp T$

Average Treatment Effect on the Treated (ATT)

ATE:

$$\mathbb{E}[Y(1) - Y(0)] = \mathbb{E}[Y \mid T = 1] - \mathbb{E}[Y \mid T = 0]$$

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Weaker Unconfoundedness:

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$= \mathbb{E}[Y \mid T = 1] - \mathbb{E}[Y \mid T = 0]$

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\end{aligned}$$

Question:
What is the difference between the ATE and the ATT?

Introducing Time

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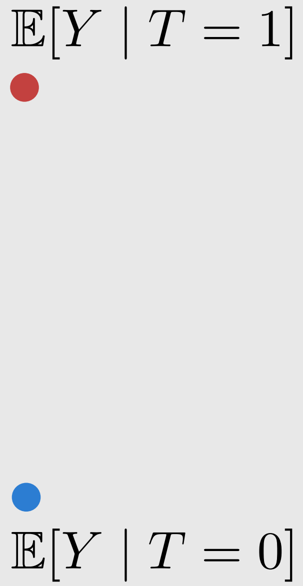

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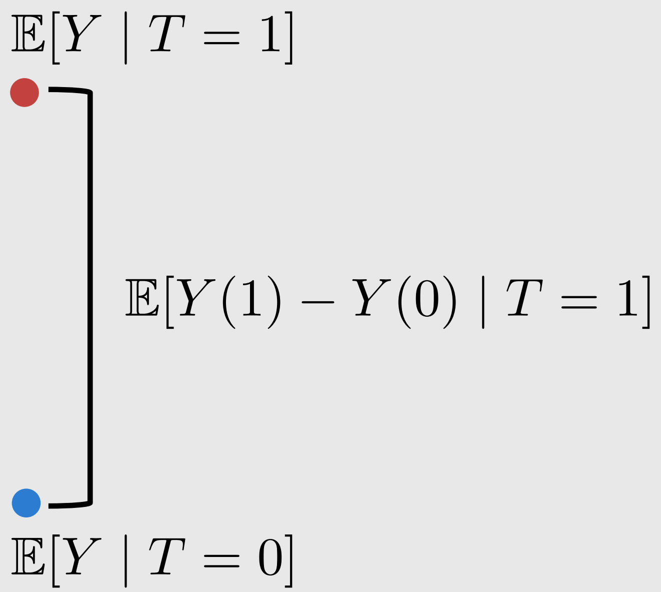

Unconfoundedness:

$Y(0) \perp\!\!\!\perp T$

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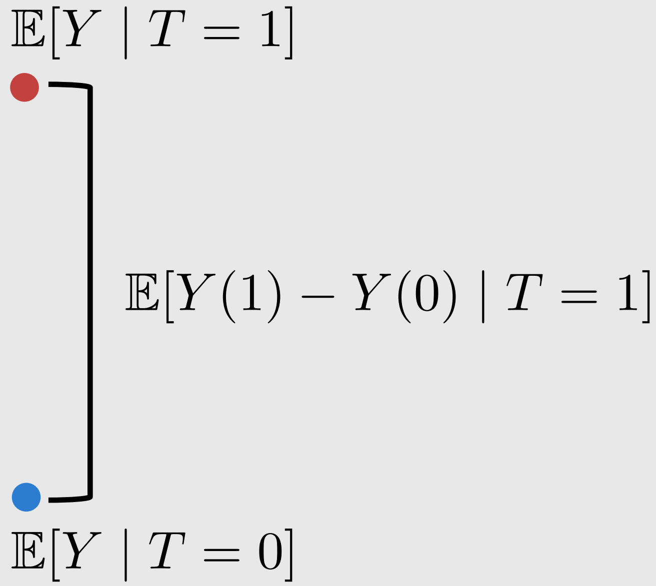

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ATT estimand without time: $\mathbb{E}[Y(1) - Y(0) \mid T = 1]$

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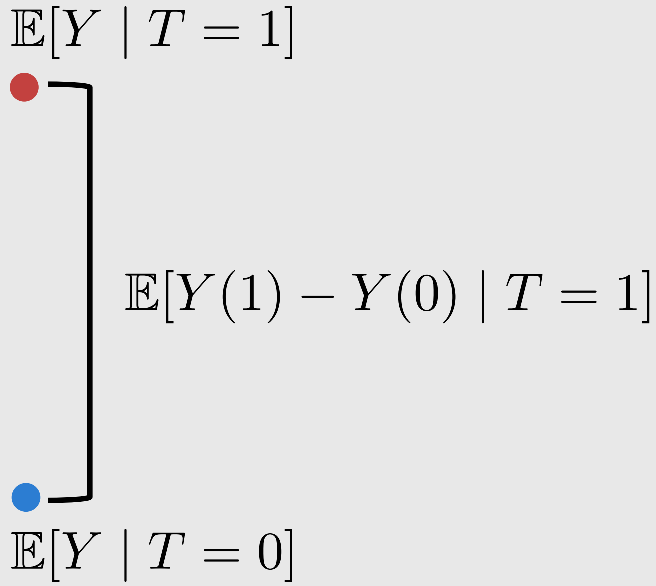

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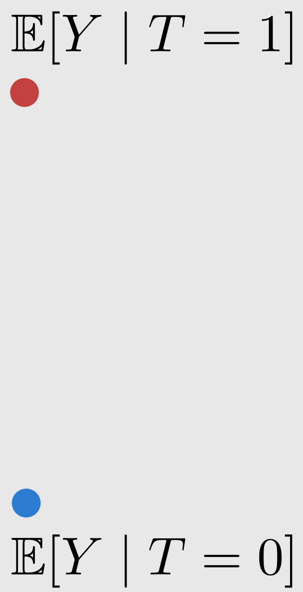

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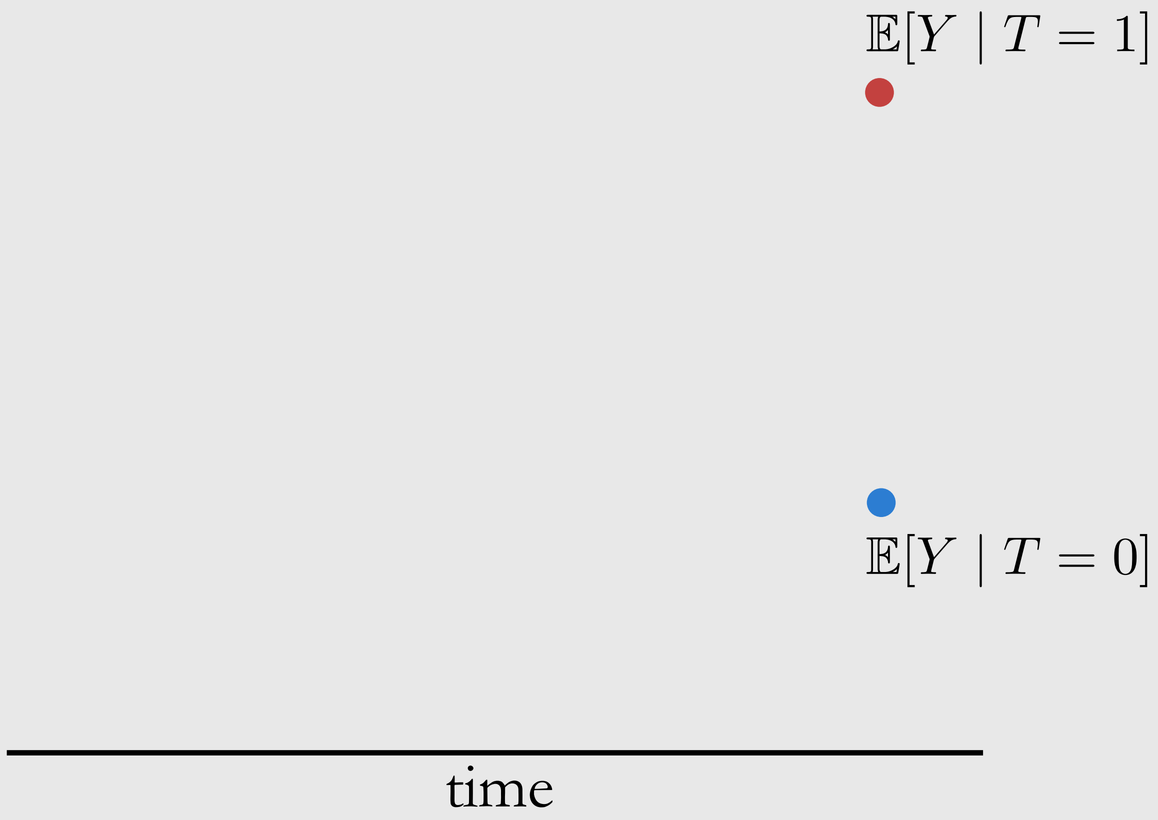

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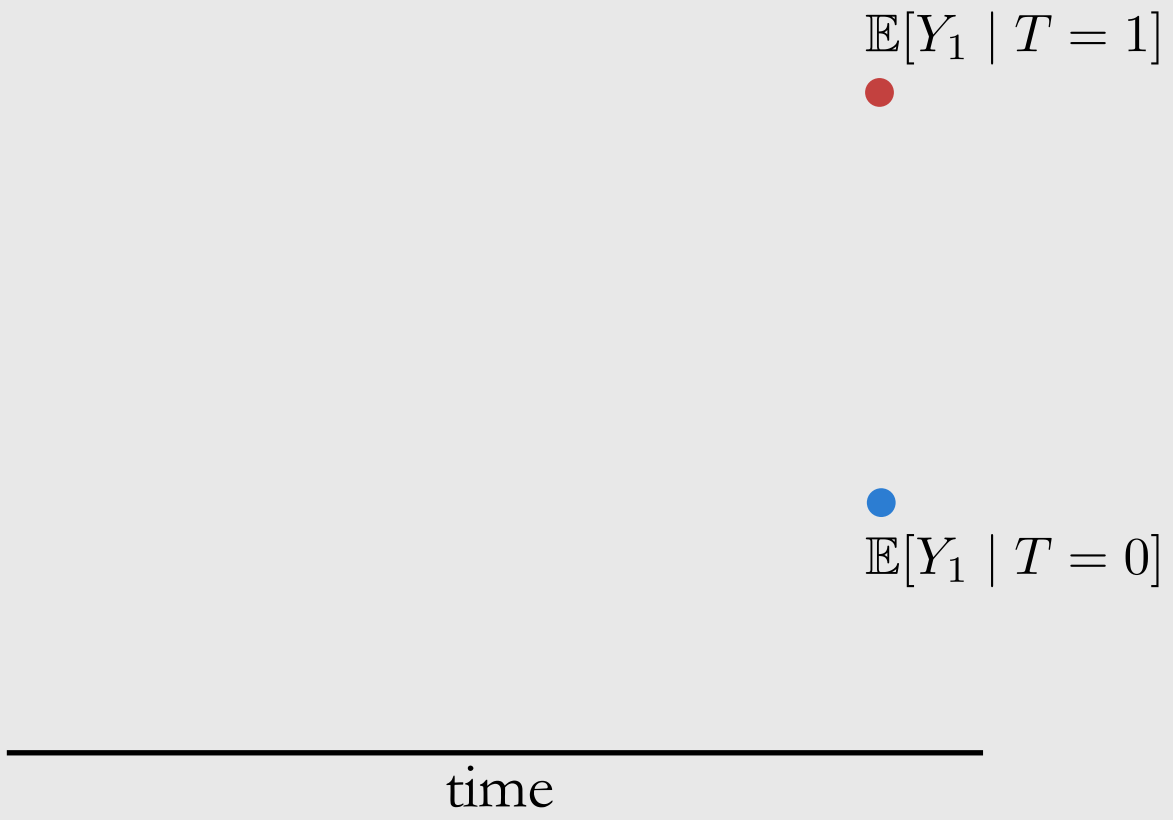

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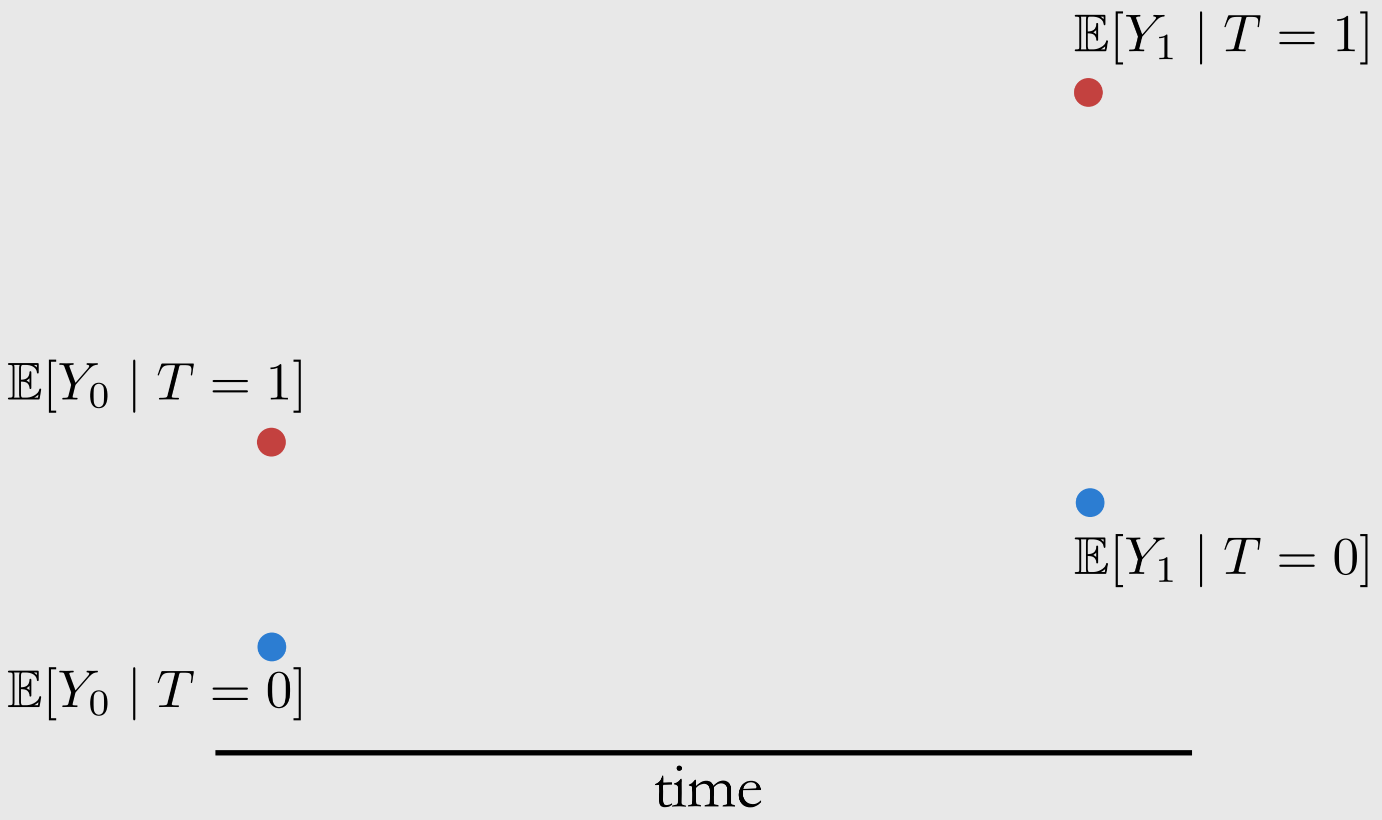

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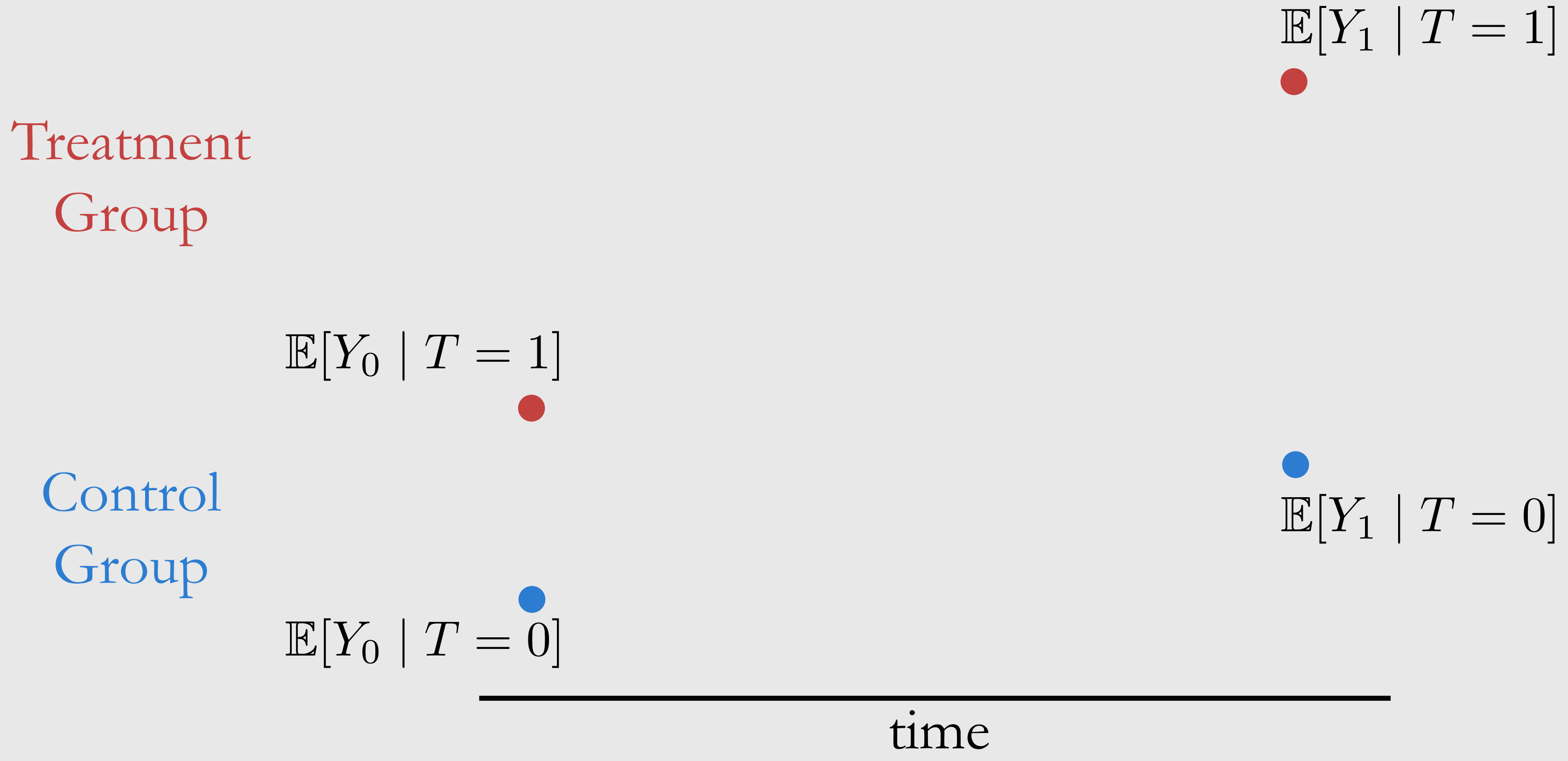

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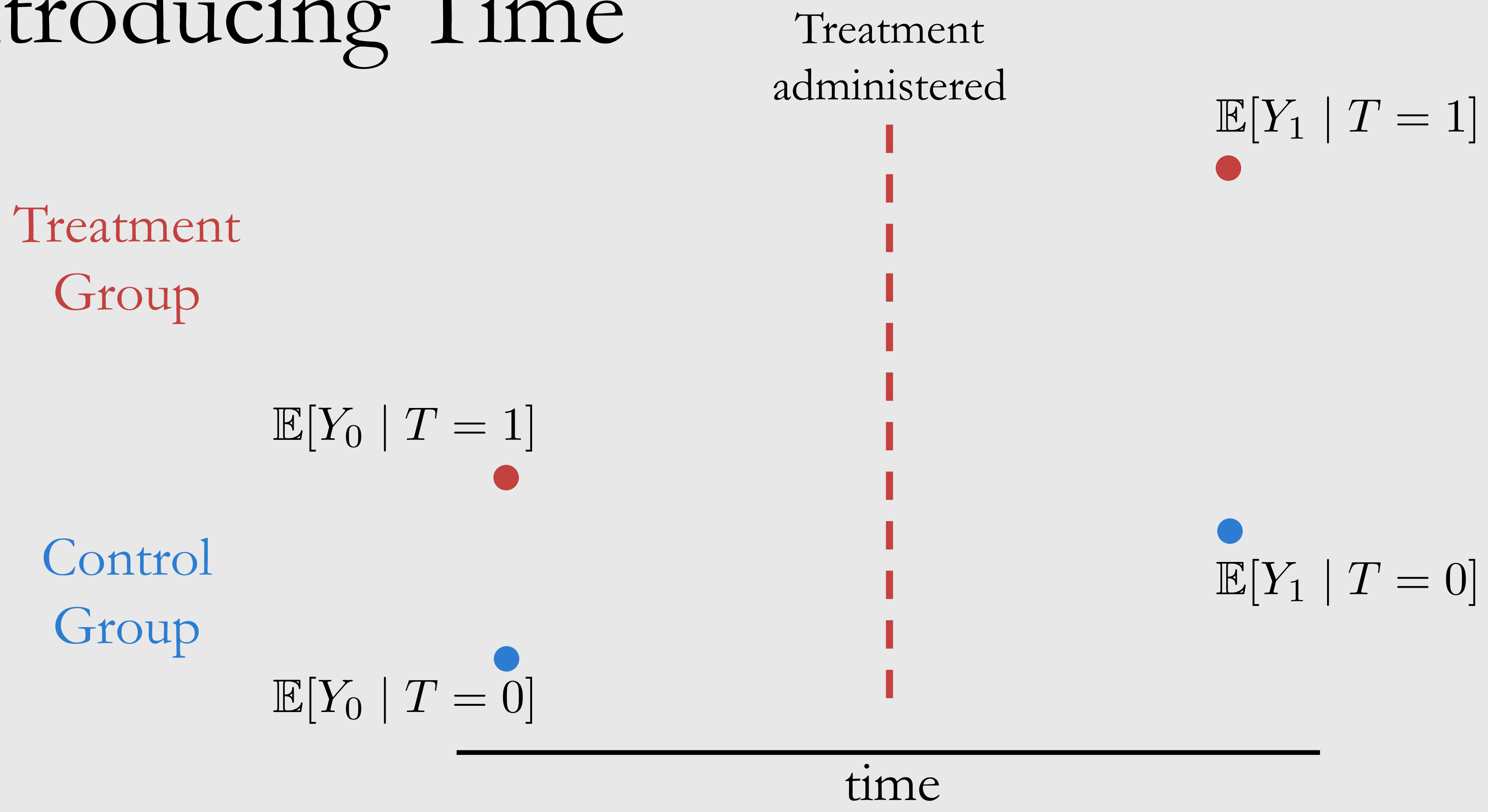

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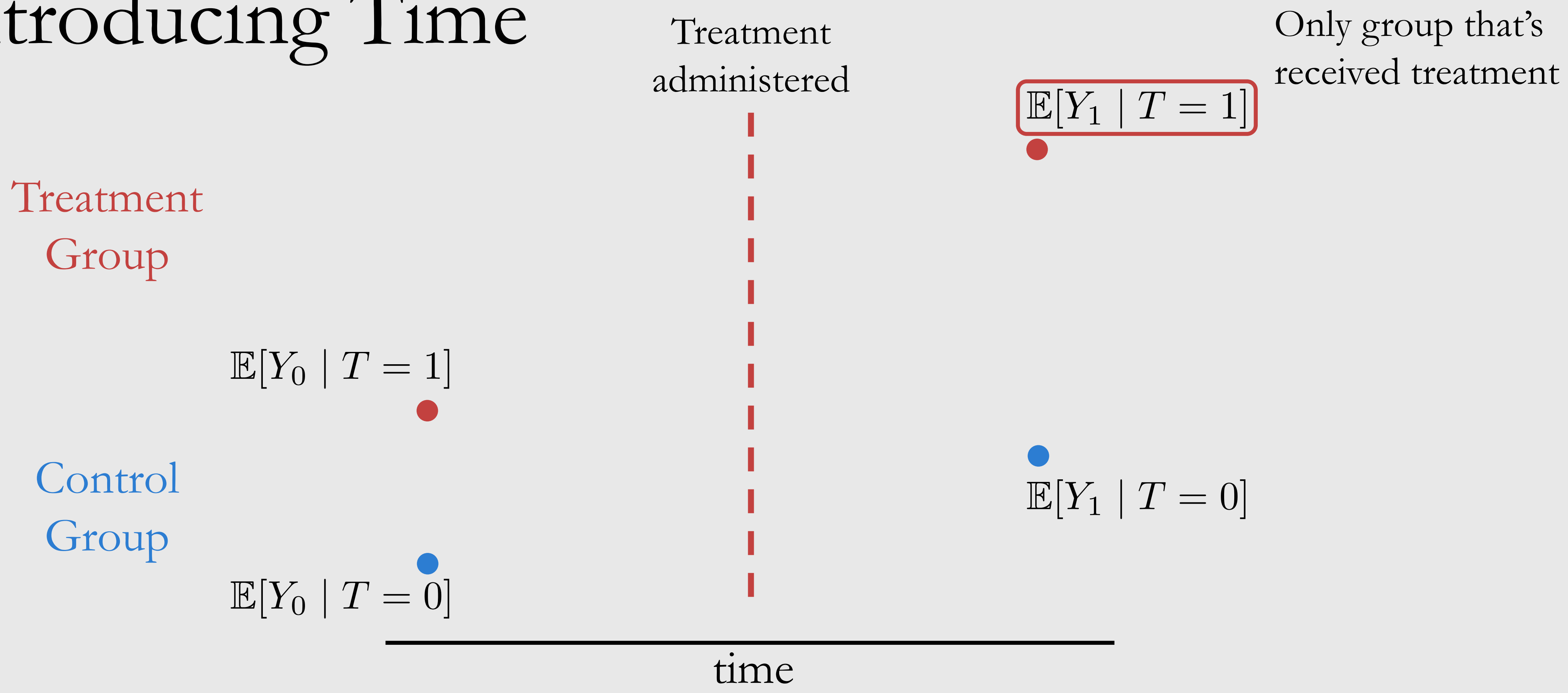

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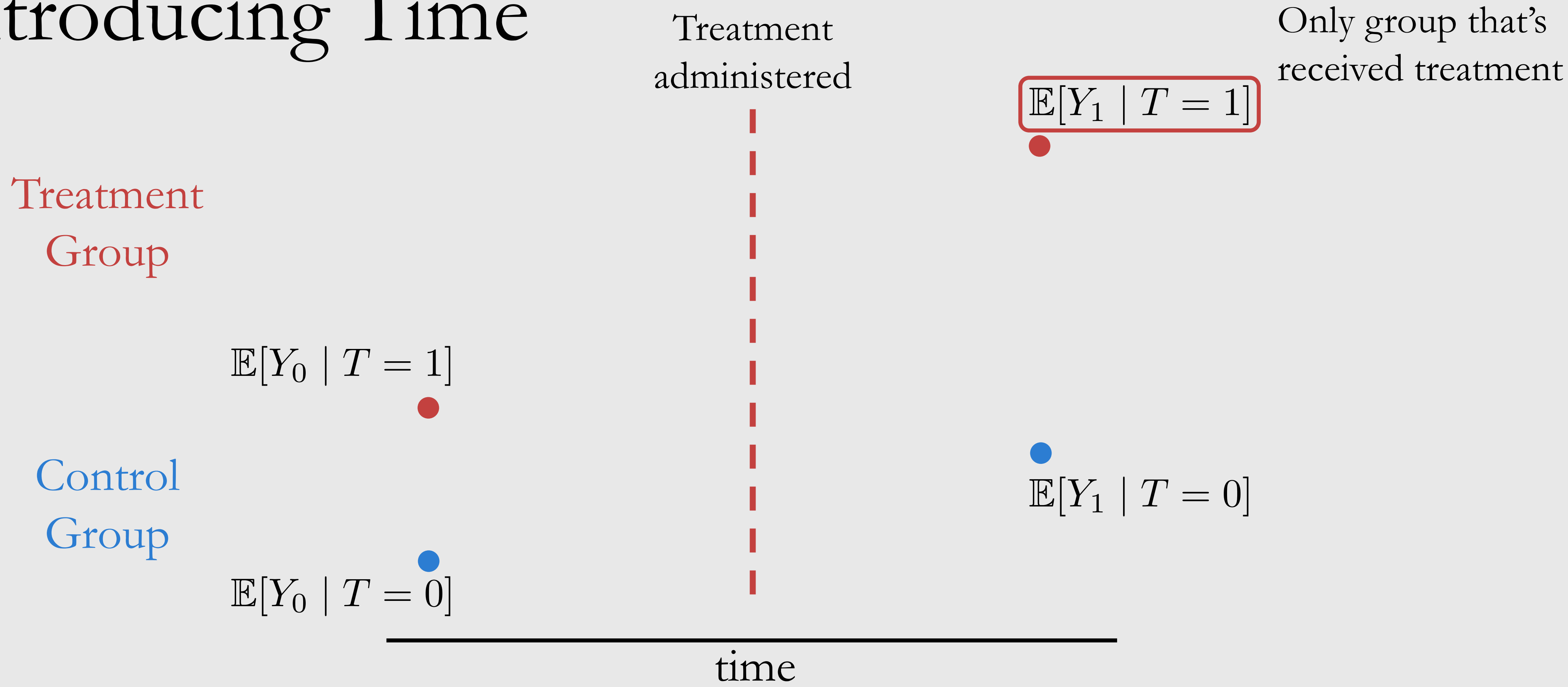

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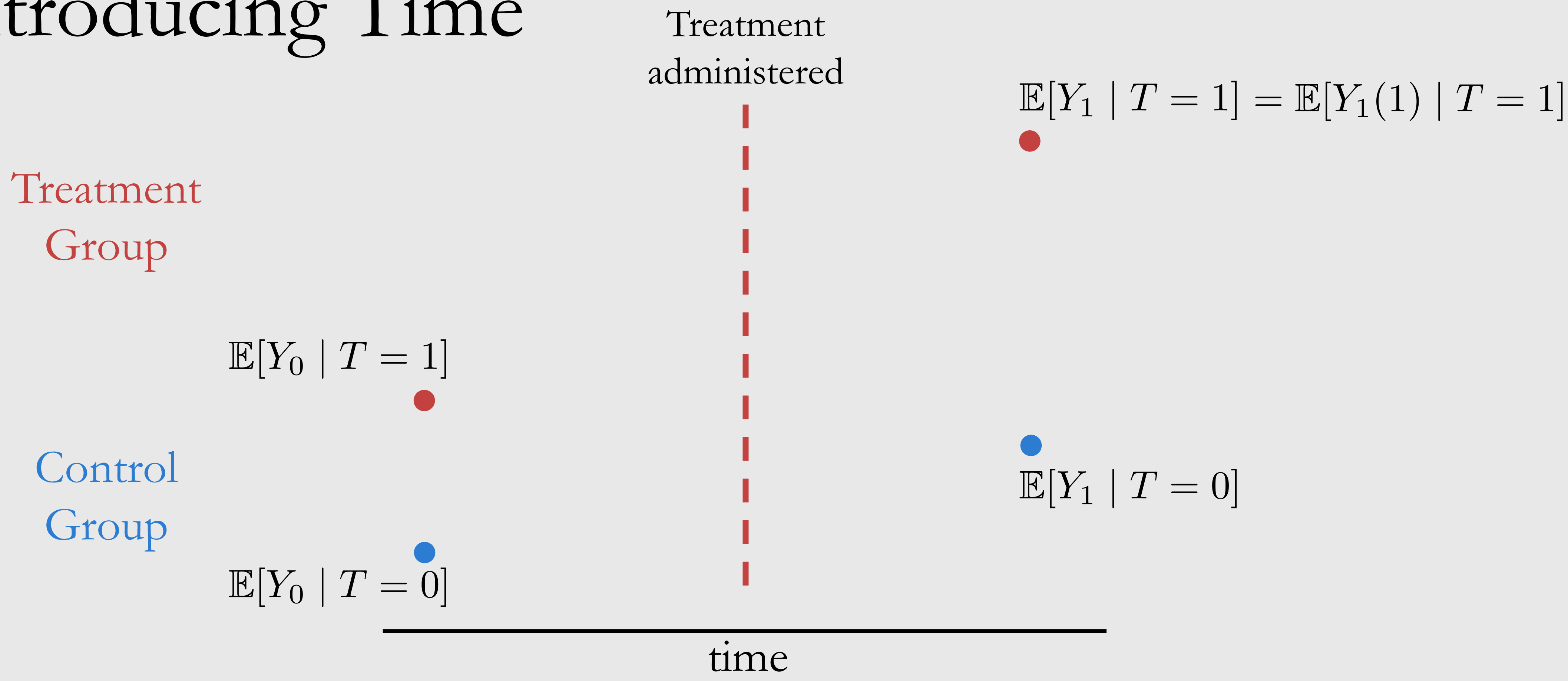

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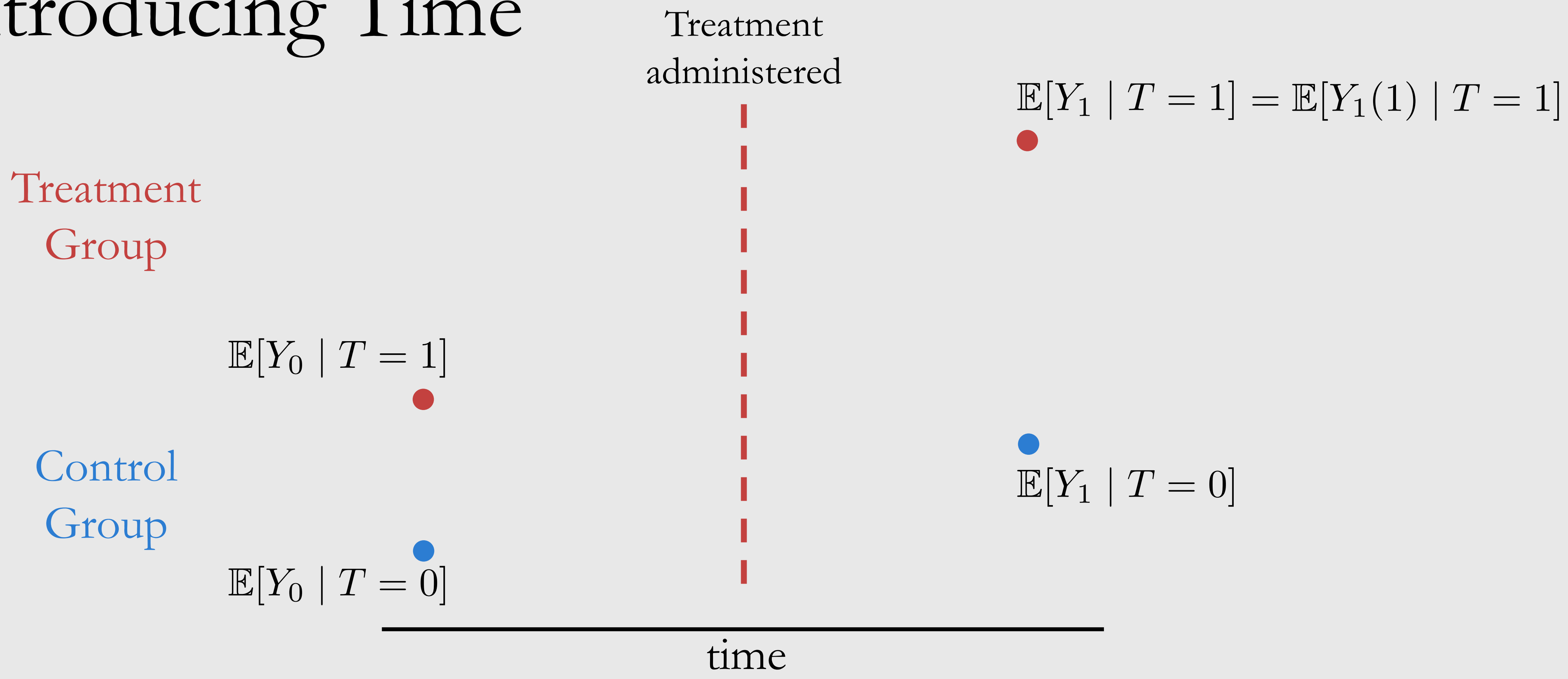

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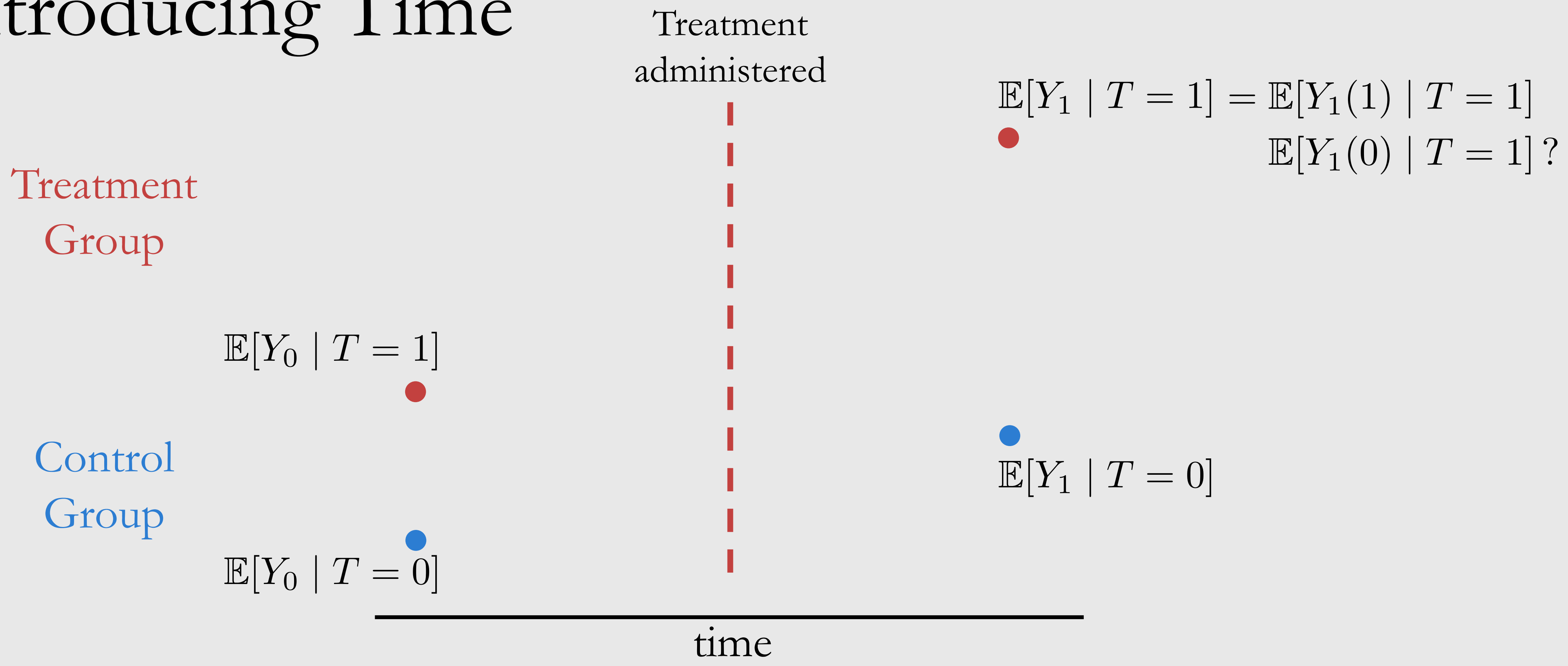

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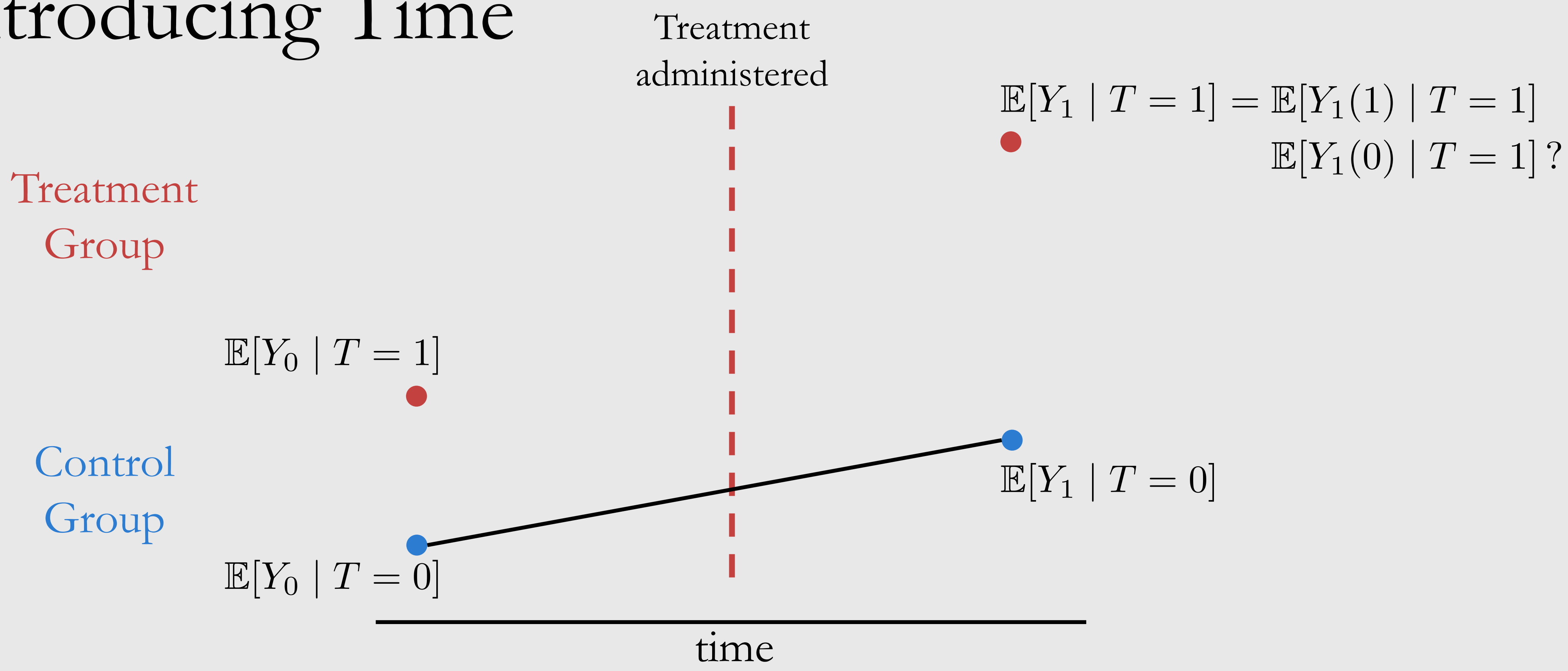

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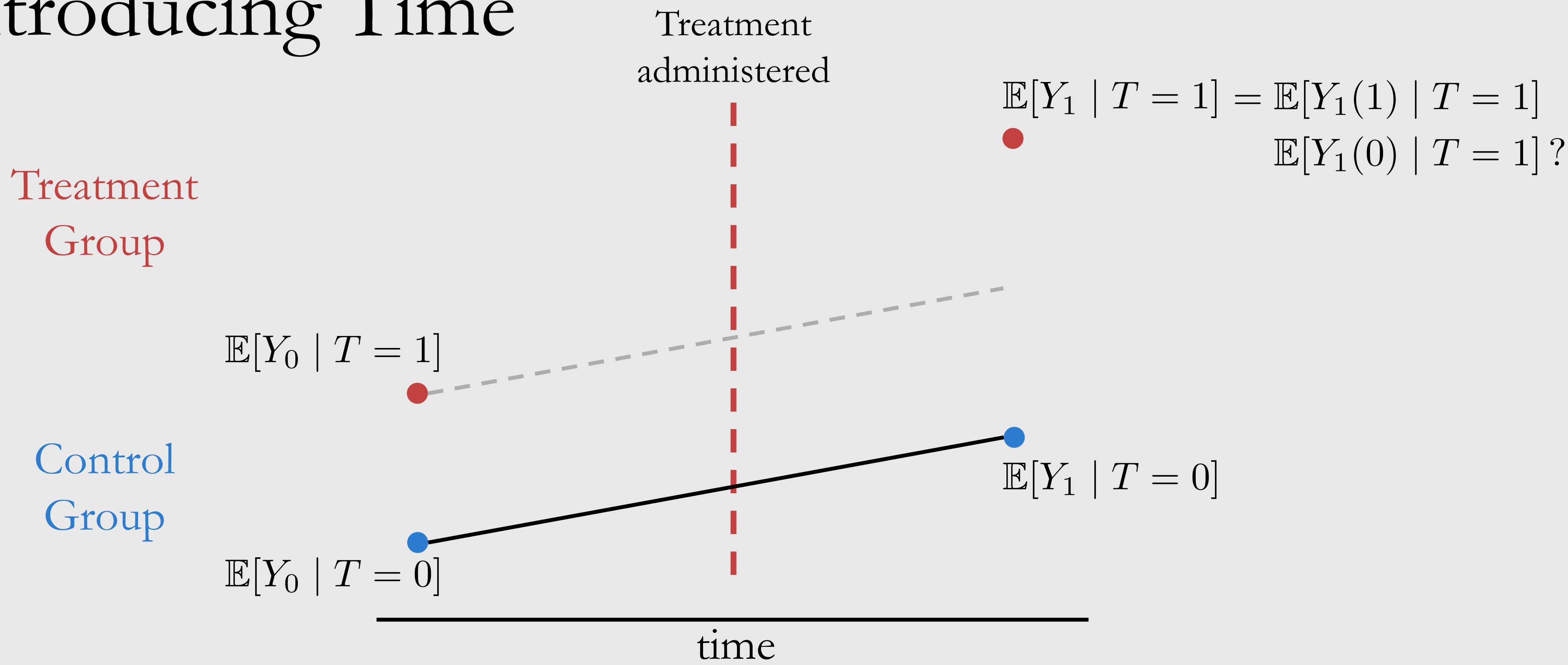

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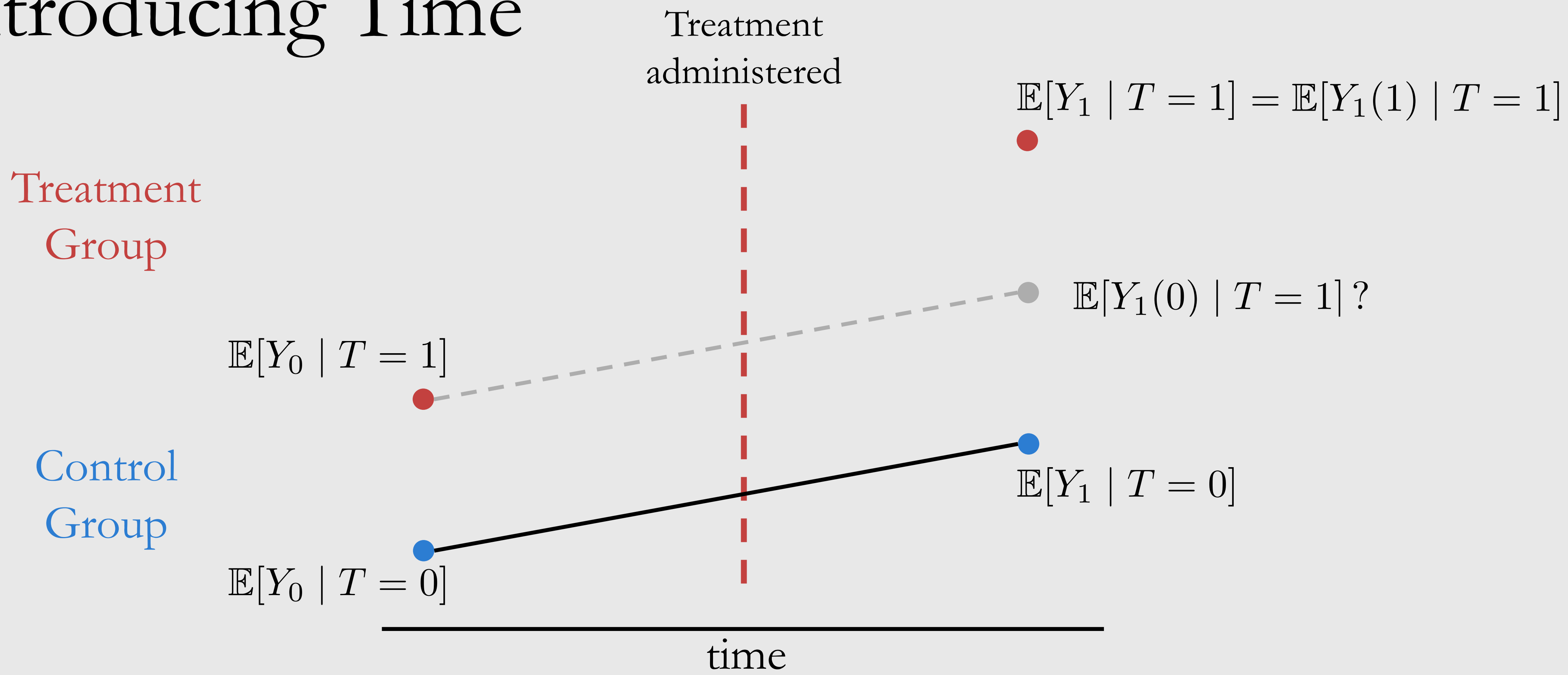

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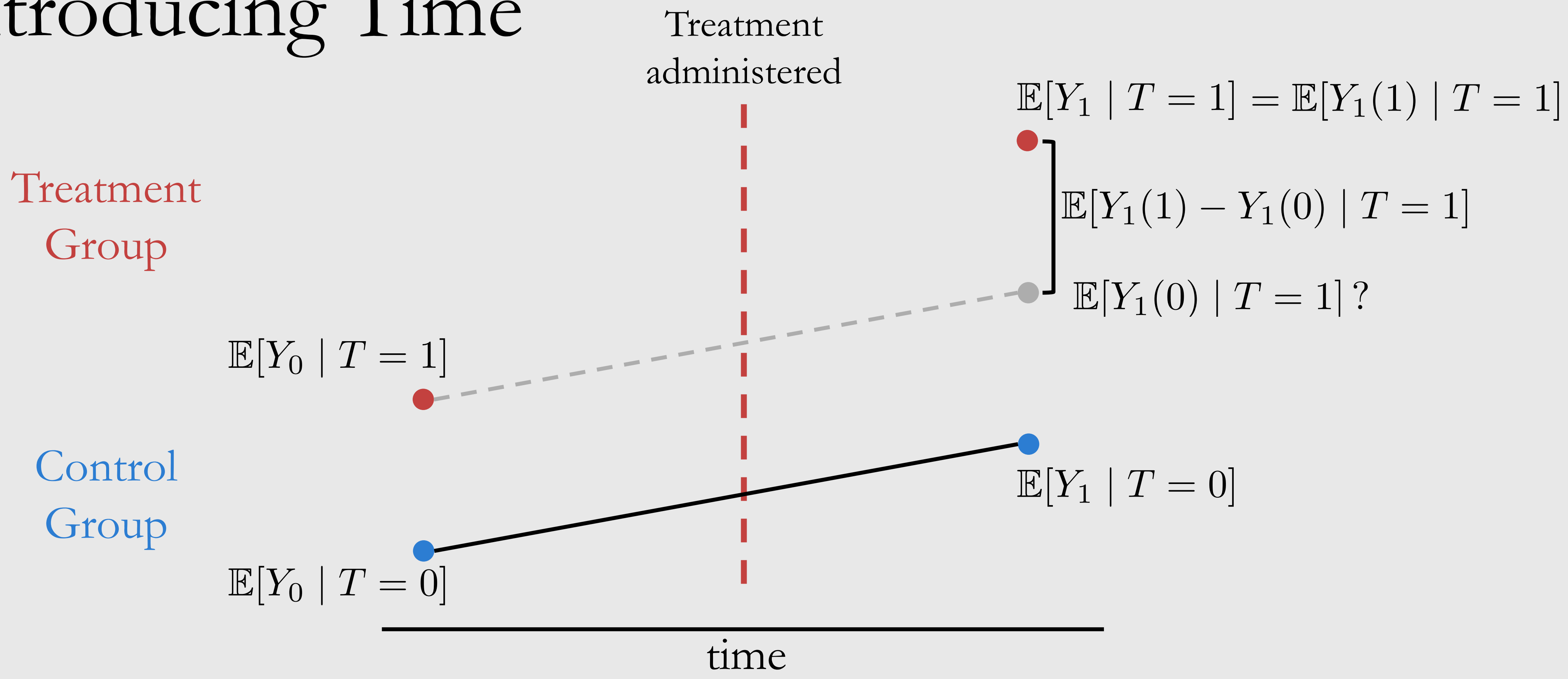

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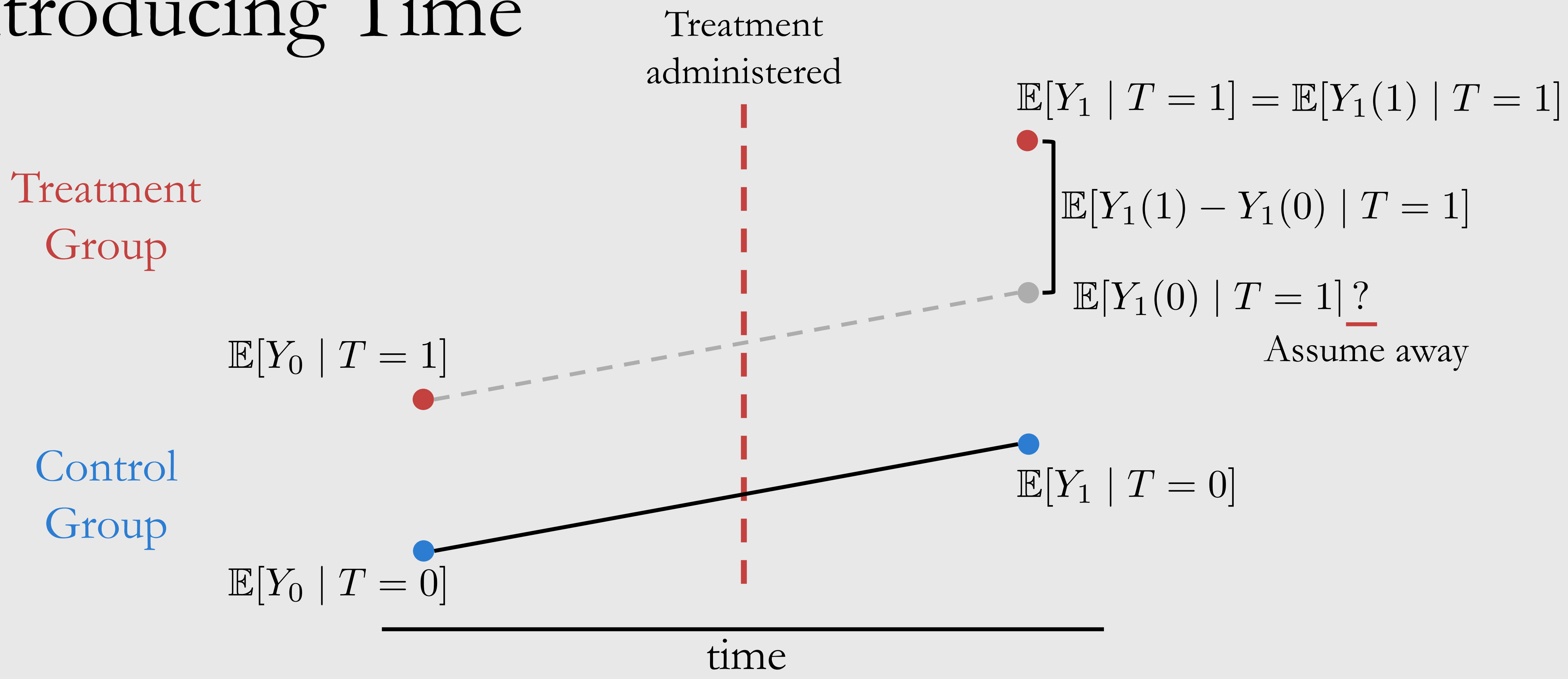

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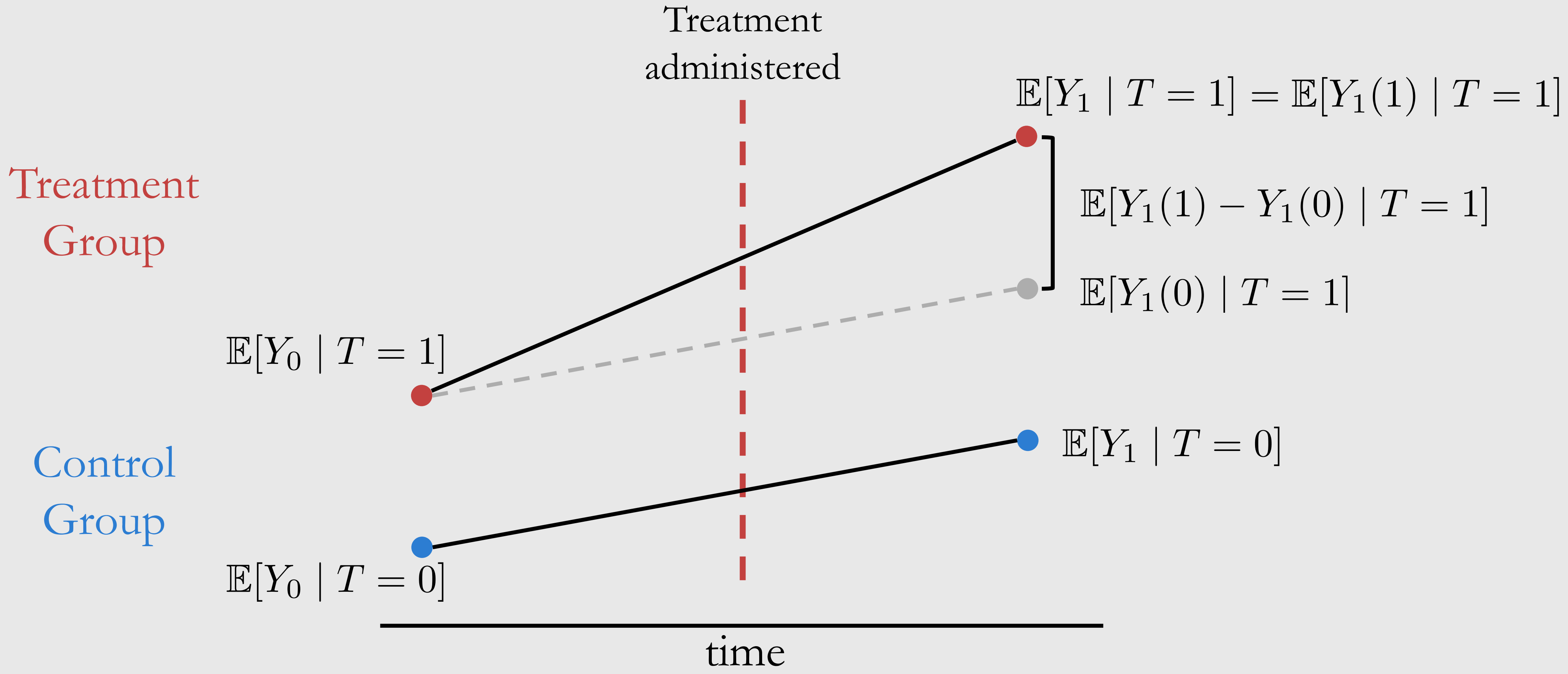
Difference-in-Differences



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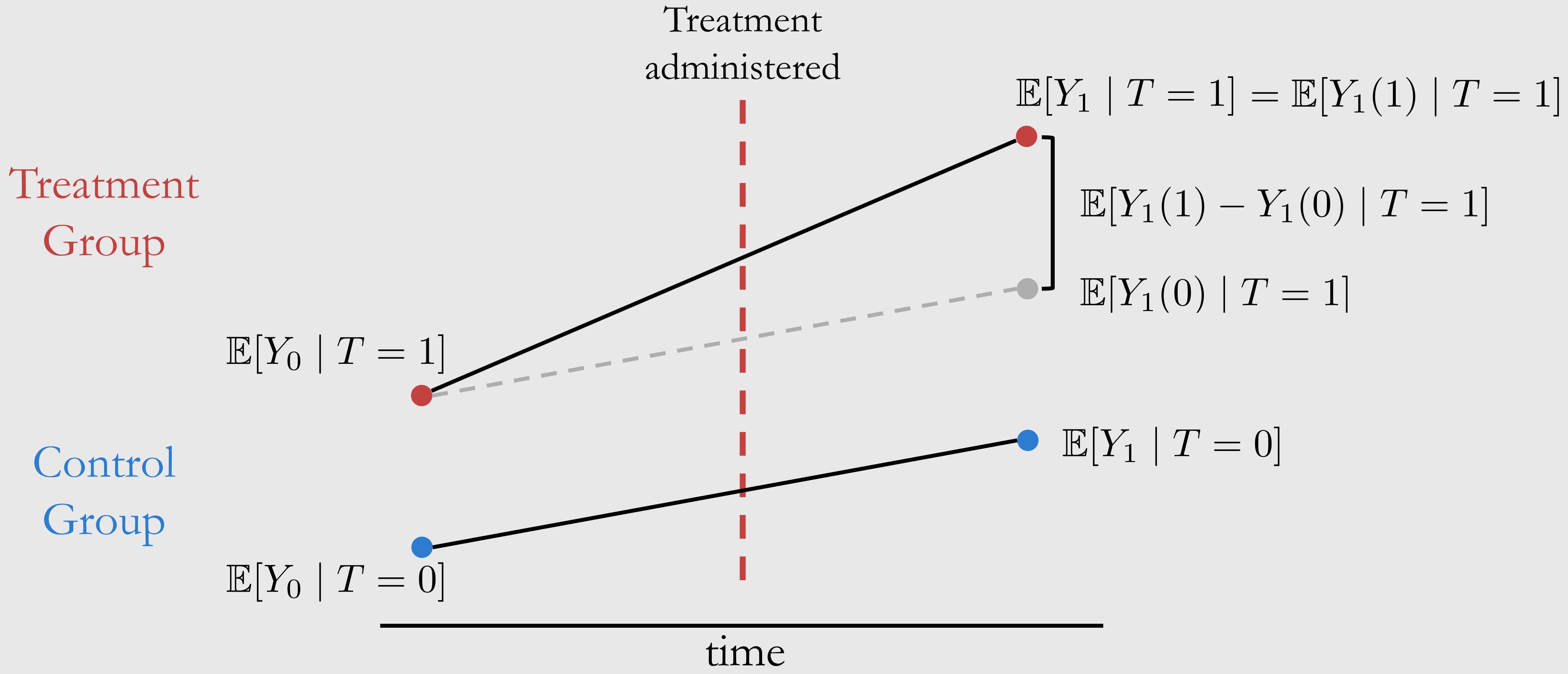

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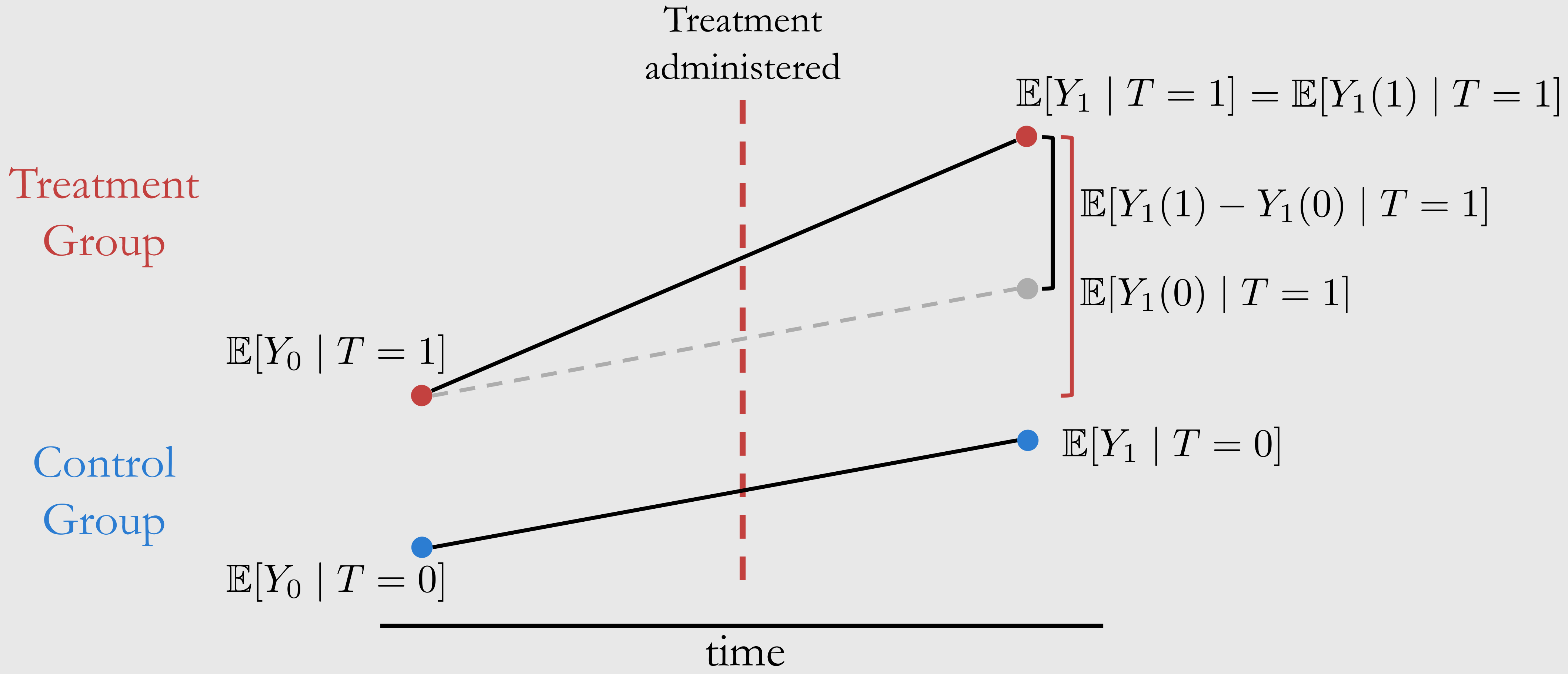

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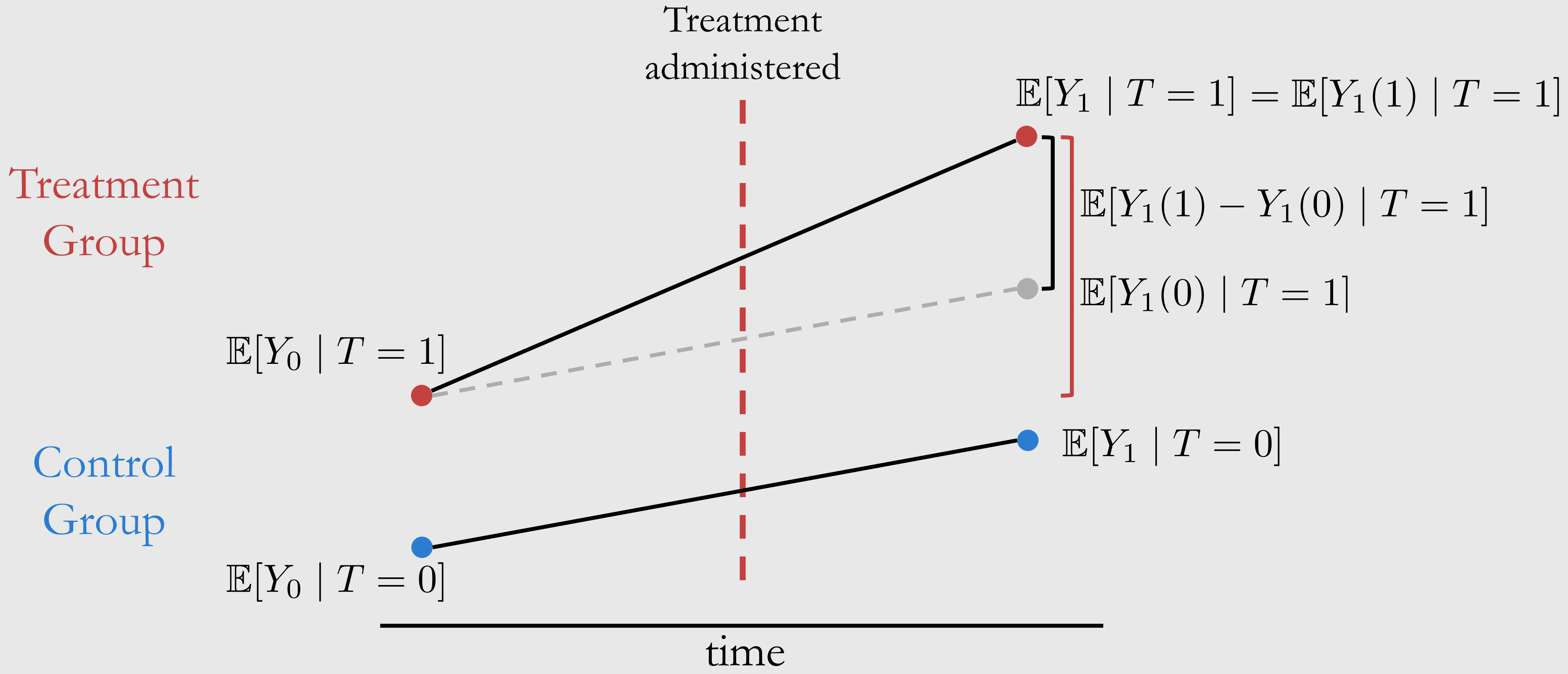

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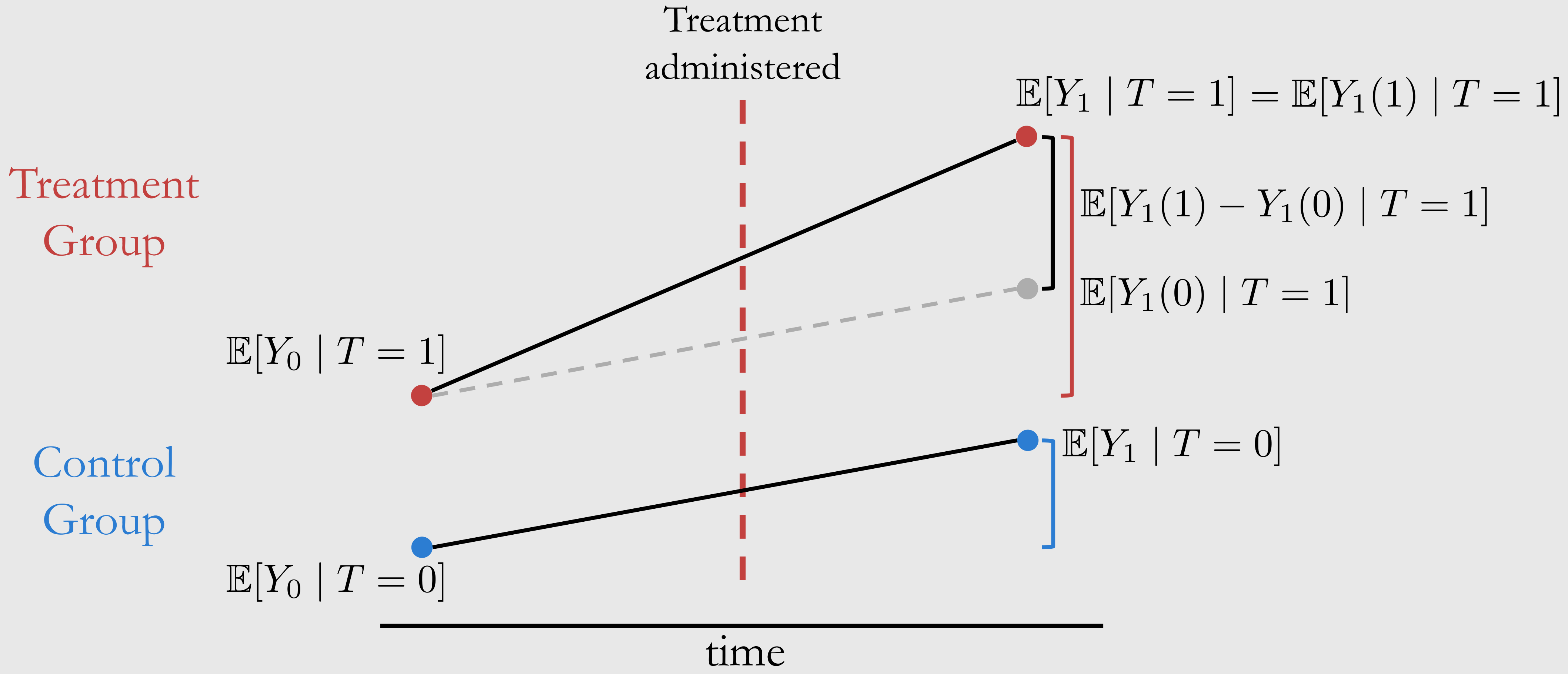

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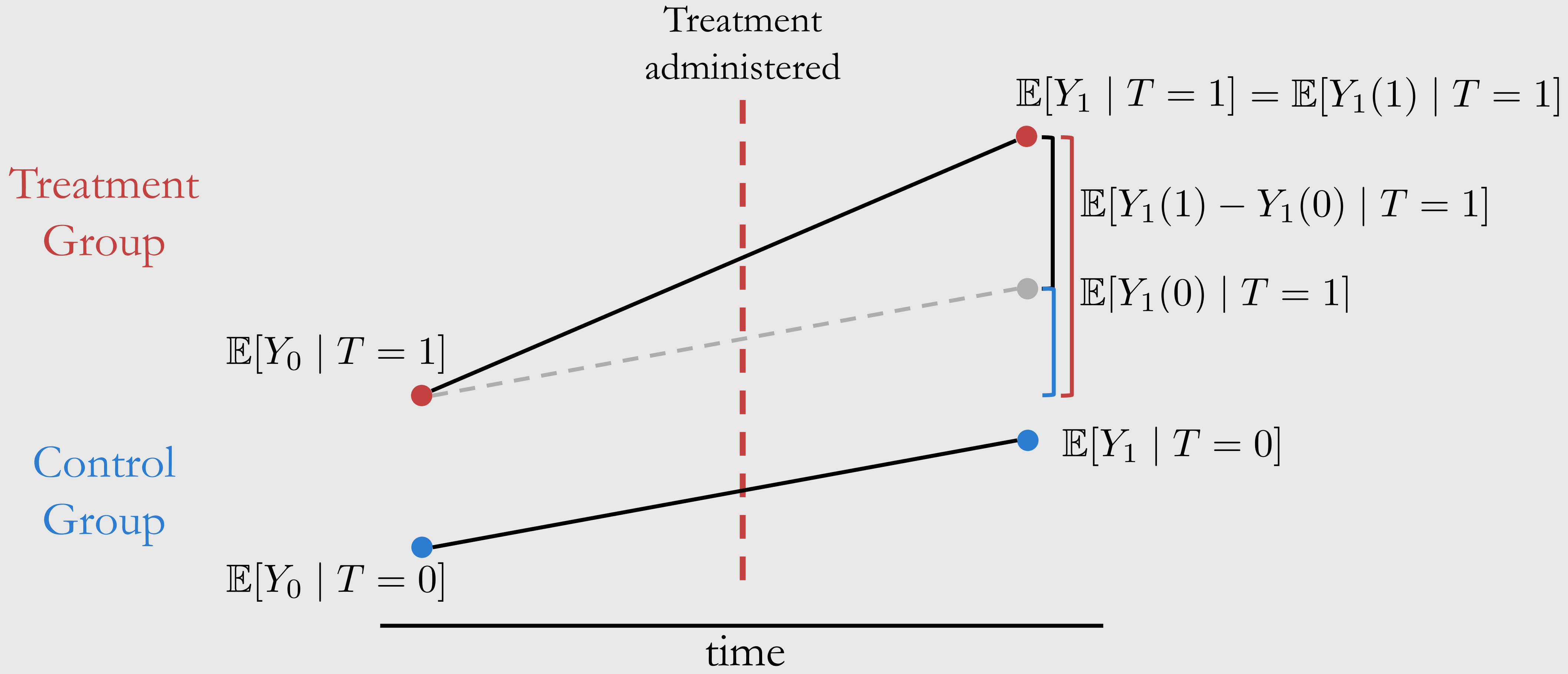

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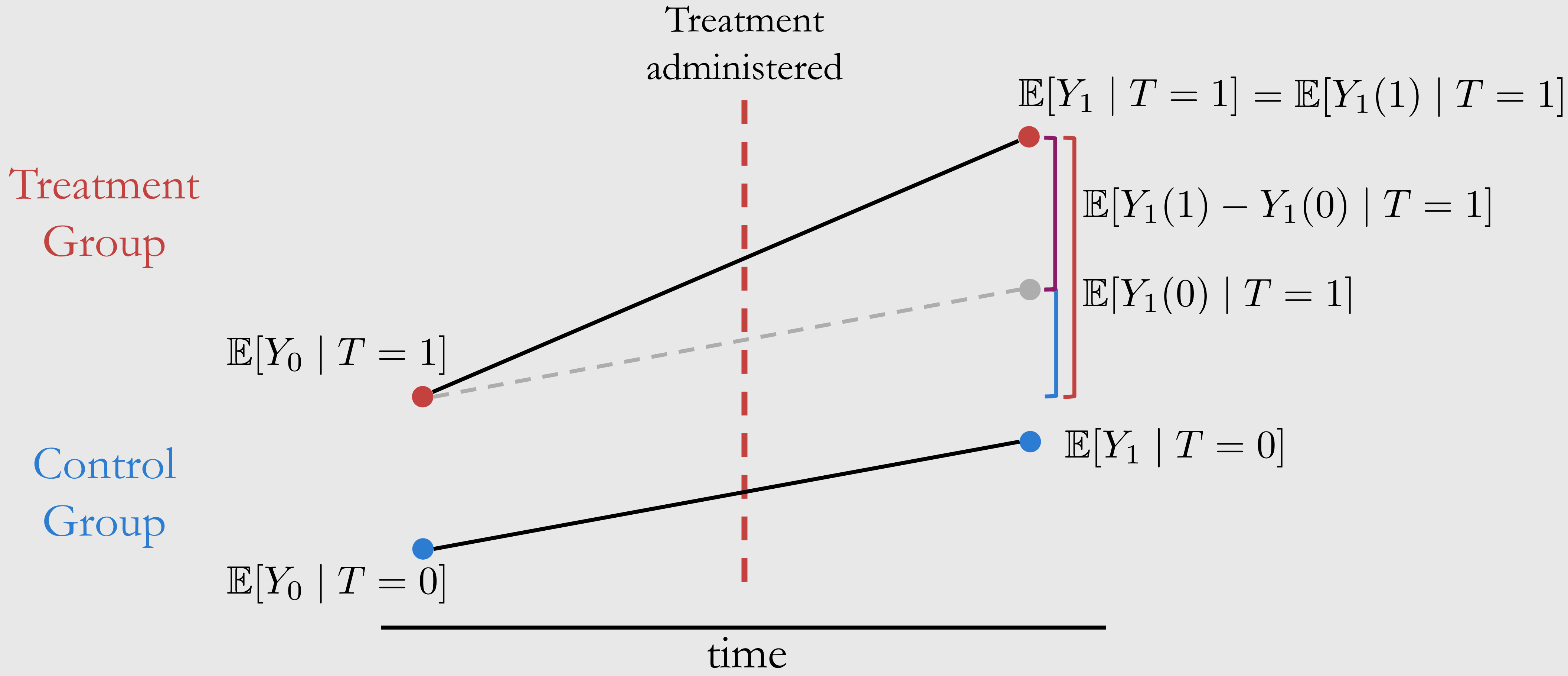

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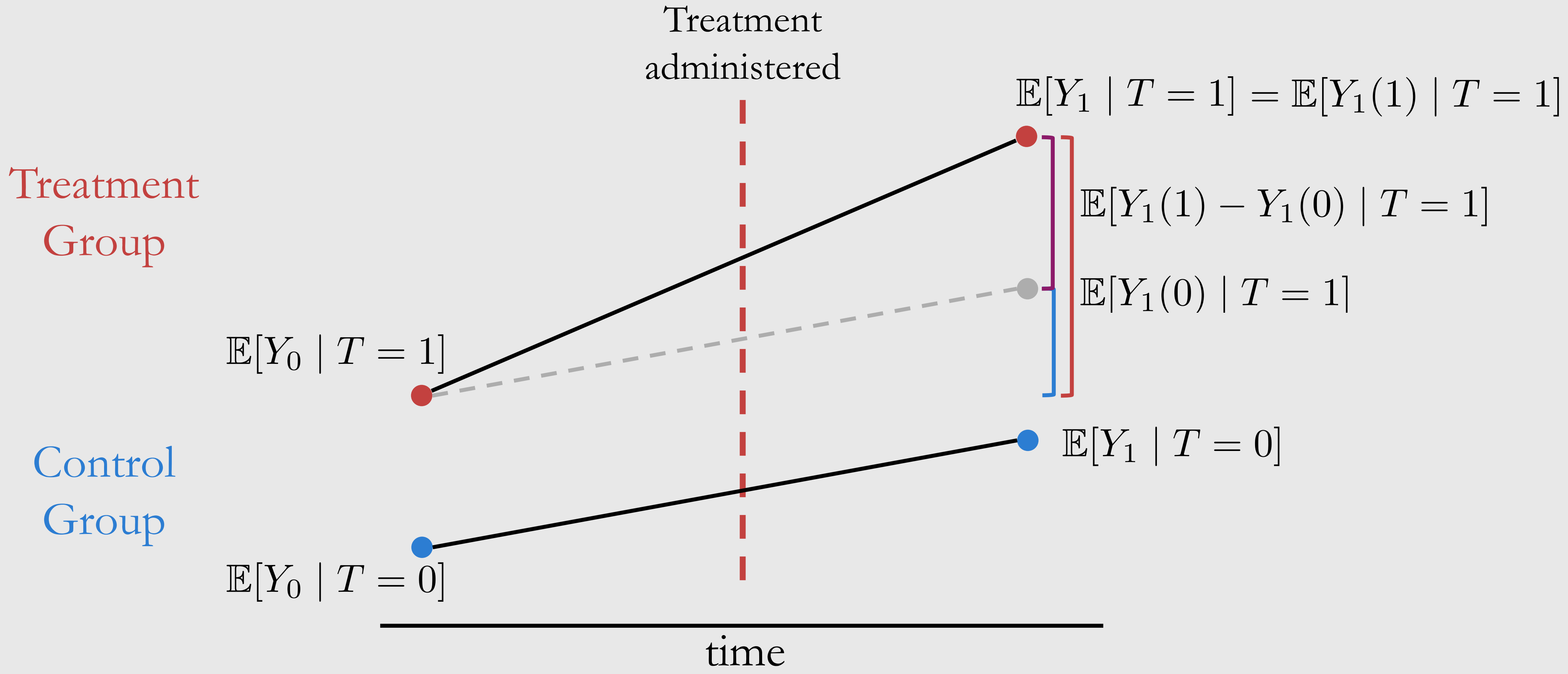

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Tolerates Time-Invariant Unobserved Confounding

Unobserved confounders that are constant with time are no problem, since they'll cancel out in the time differences

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = \underbrace{(\mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_0 \mid T = 1])}_{\text{Time difference in treatment group}} - \underbrace{(\mathbb{E}[Y_1 \mid T = 0] - \mathbb{E}[Y_0 \mid T = 0])}_{\text{Time difference in control group}}$$

Question:
How would you estimate the terms on the right-hand side of the difference-in-differences equation?

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = (\mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_0 \mid T = 1]) - (\mathbb{E}[Y_1 \mid T = 0] - \mathbb{E}[Y_0 \mid T = 0])$$

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Consistency Assumption Extended to Time

$$\forall \tau, \quad T = t \implies Y_\tau = Y_\tau(t)$$

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Causal estimand Statistical estimand

Examples: $\mathbb{E}[Y_\tau(1) \mid T = 1] = \mathbb{E}[Y_\tau \mid T = 1]$

Consistency Assumption Extended to Time

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Causal estimand | Statistical estimand | Causal estimand | Statistical estimand

Examples: $\mathbb{E}[Y_\tau(1) \mid T = 1] = \mathbb{E}[Y_\tau \mid T = 1] \quad \mathbb{E}[Y_\tau(0) \mid T = 0] = \mathbb{E}[Y_\tau \mid T = 0]$

Consistency Assumption Extended to Time

$$\forall \tau, \quad T = t \implies Y_\tau = Y_\tau(t)$$

	Causal estimand		Statistical estimand	Causal estimand		Statistical estimand
Examples:	$\mathbb{E}[Y_\tau(1) \mid T = 1]$	$=$	$\mathbb{E}[Y_\tau \mid T = 1]$	$\mathbb{E}[Y_\tau(0) \mid T = 0]$	$=$	$\mathbb{E}[Y_\tau \mid T = 0]$

Counterfactual Quantities: $\mathbb{E}[Y_\tau(1) \mid T = 0]$

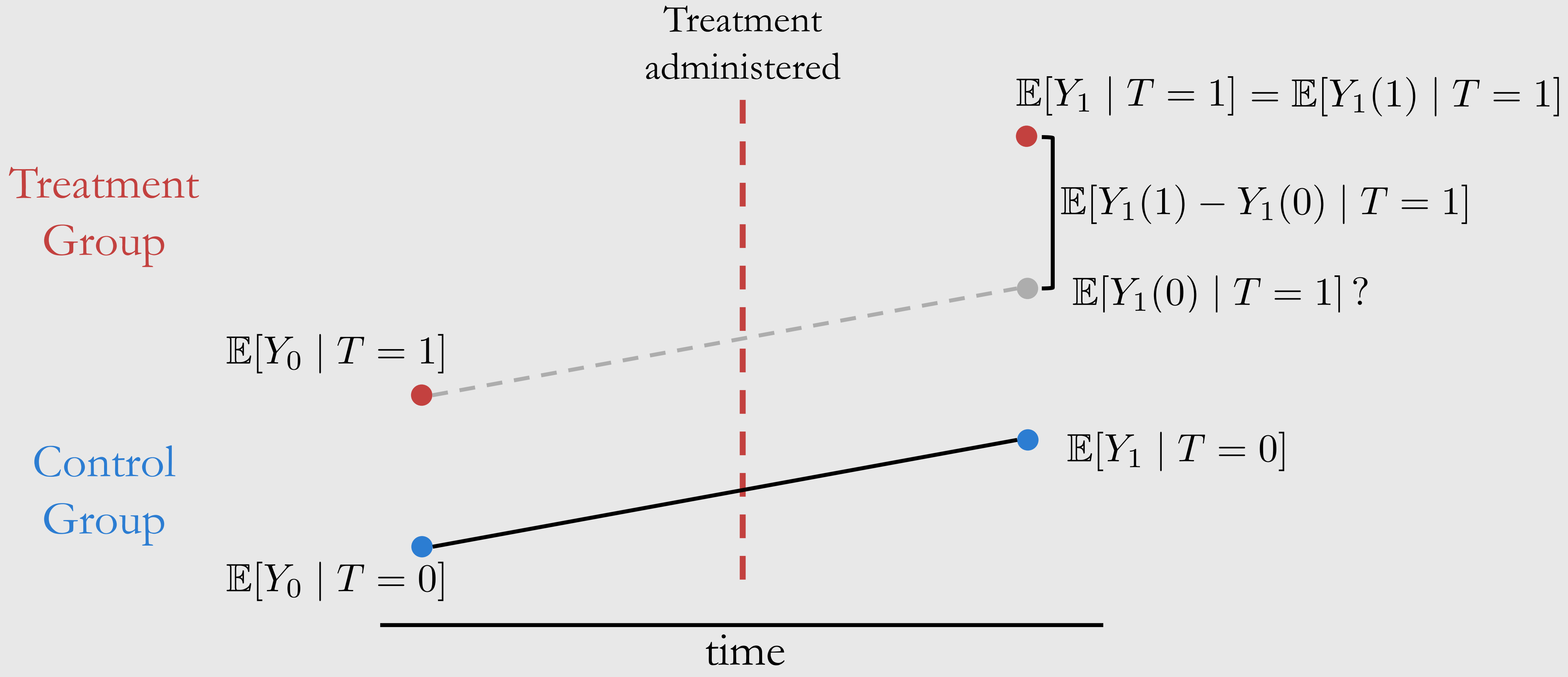
Consistency Assumption Extended to Time

$$\forall \tau, \quad T = t \implies Y_\tau = Y_\tau(t)$$

	Causal estimand		Statistical estimand	Causal estimand		Statistical estimand
Examples:	$\mathbb{E}[Y_\tau(1) \mid T = 1]$	$=$	$\mathbb{E}[Y_\tau \mid T = 1]$	$\mathbb{E}[Y_\tau(0) \mid T = 0]$	$=$	$\mathbb{E}[Y_\tau \mid T = 0]$

Counterfactual Quantities: $\mathbb{E}[Y_\tau(1) \mid T = 0]$ and $\mathbb{E}[Y_\tau(0) \mid T = 1]$

Parallel Trends Assumption



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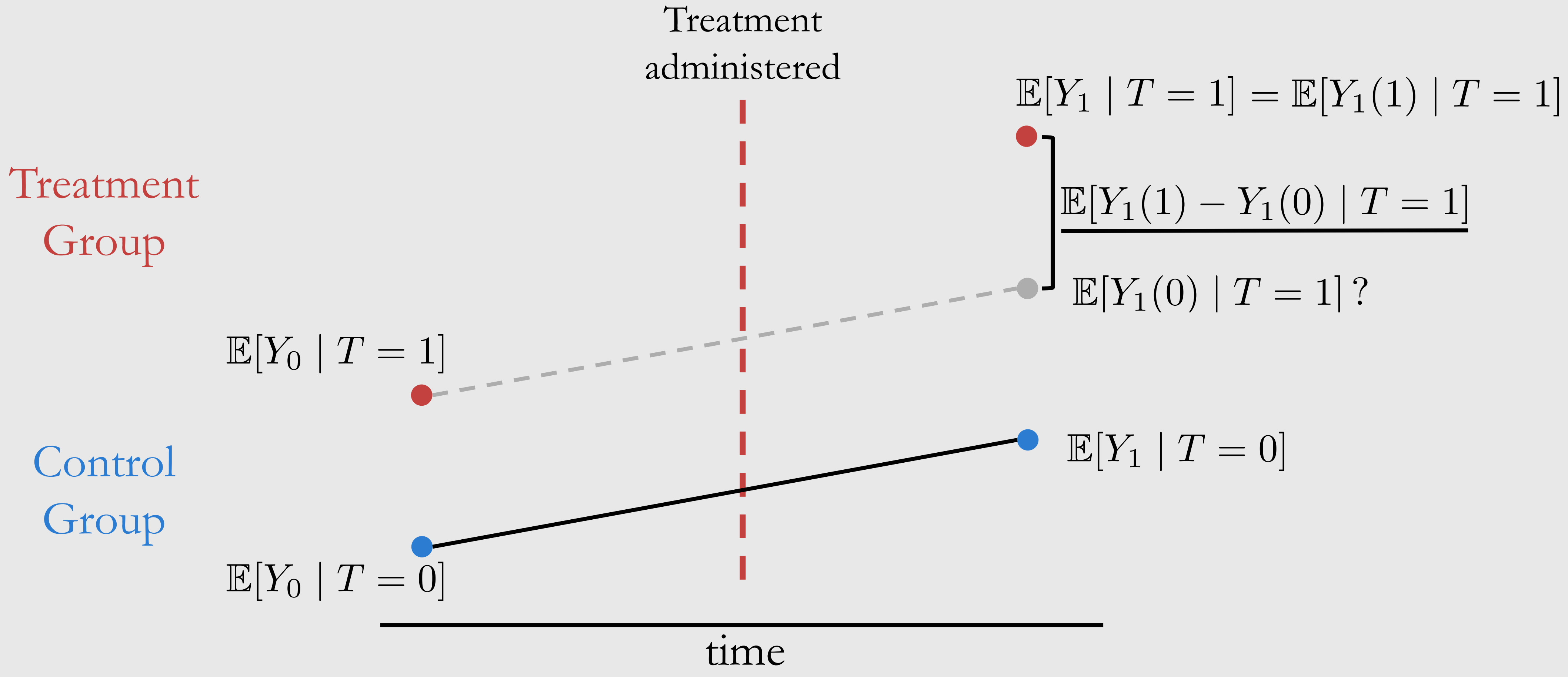

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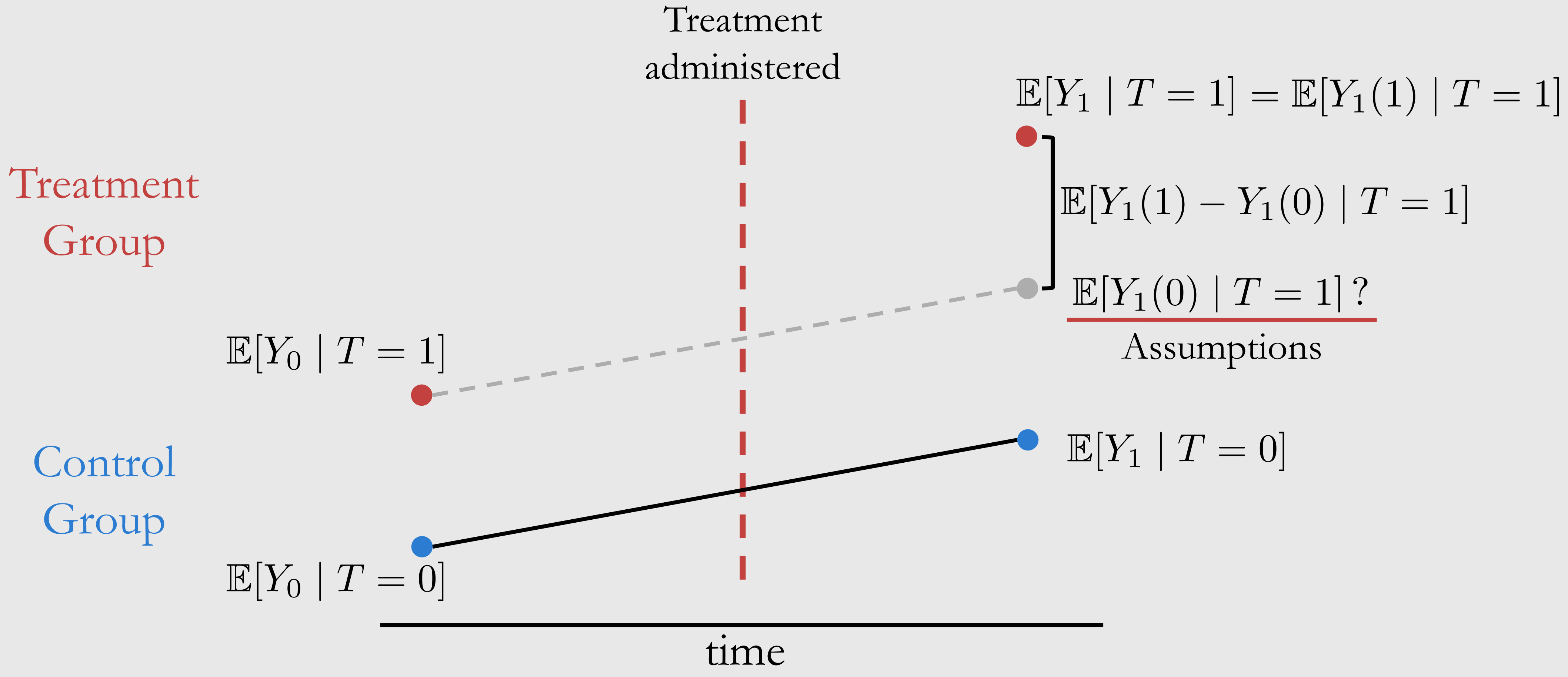
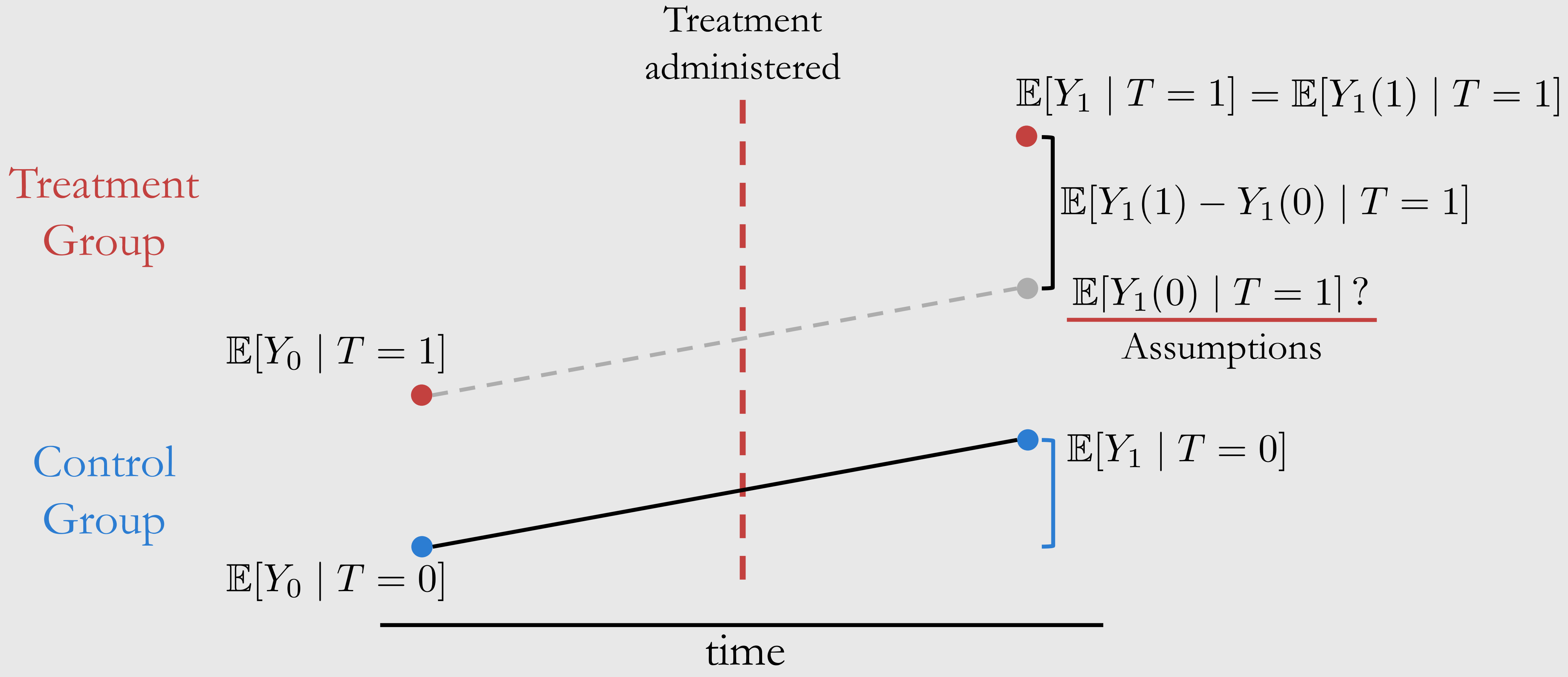
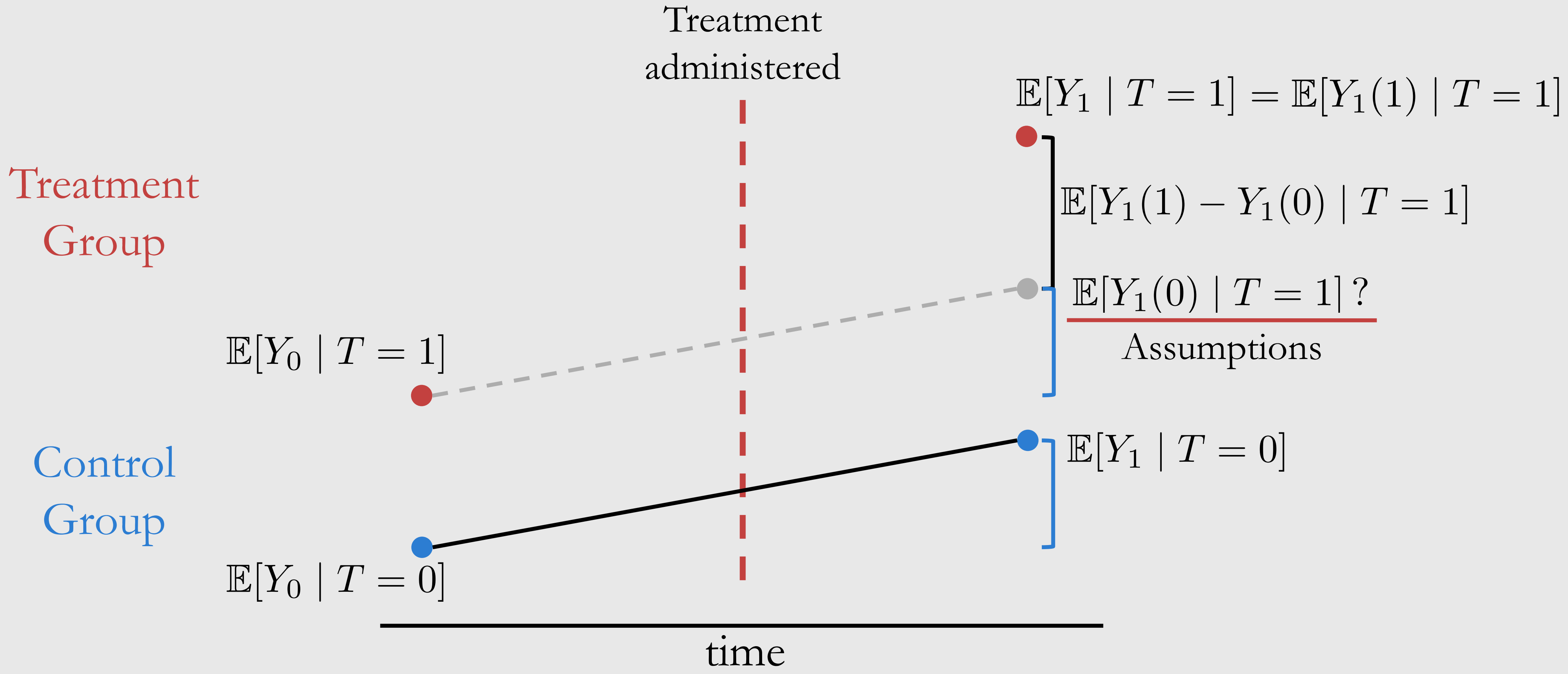


Parallel Trends Assumption



Parallel Trends Assumption



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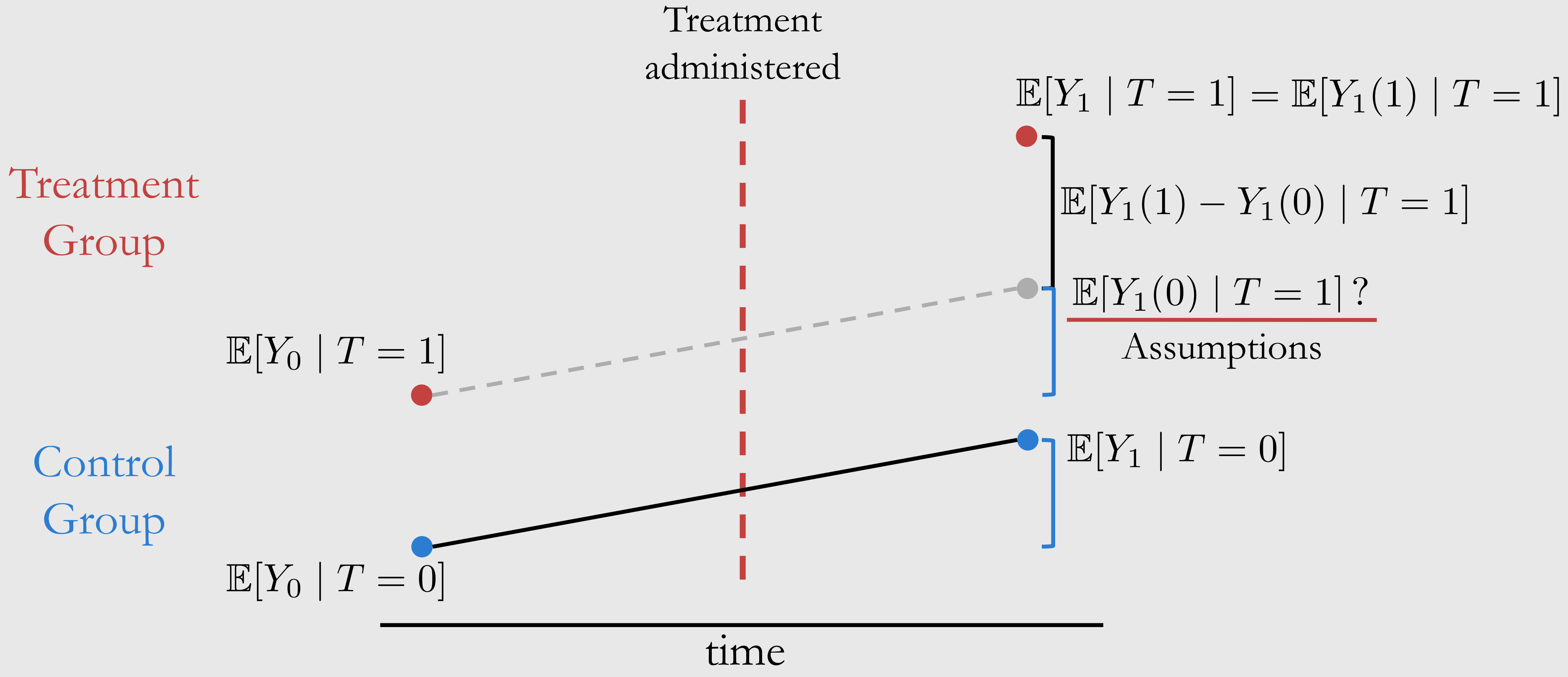

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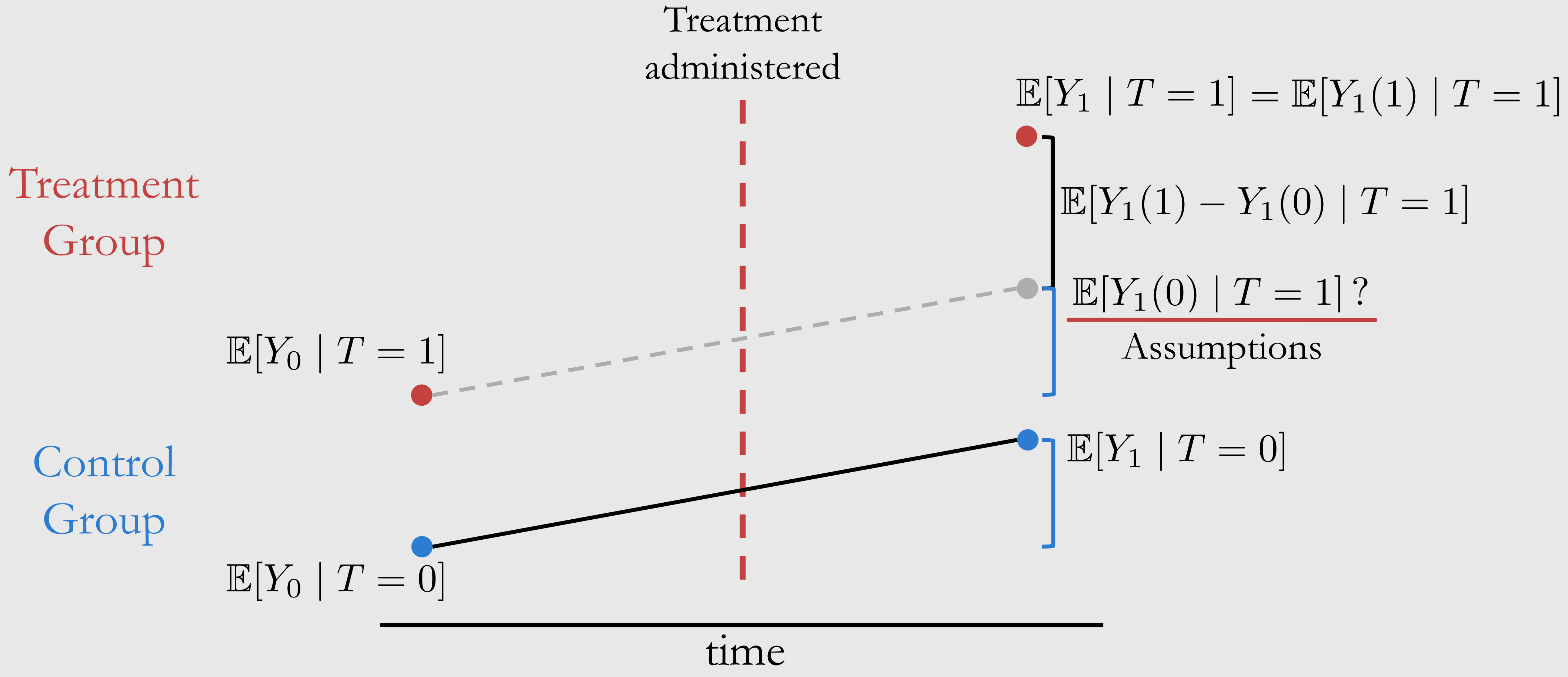
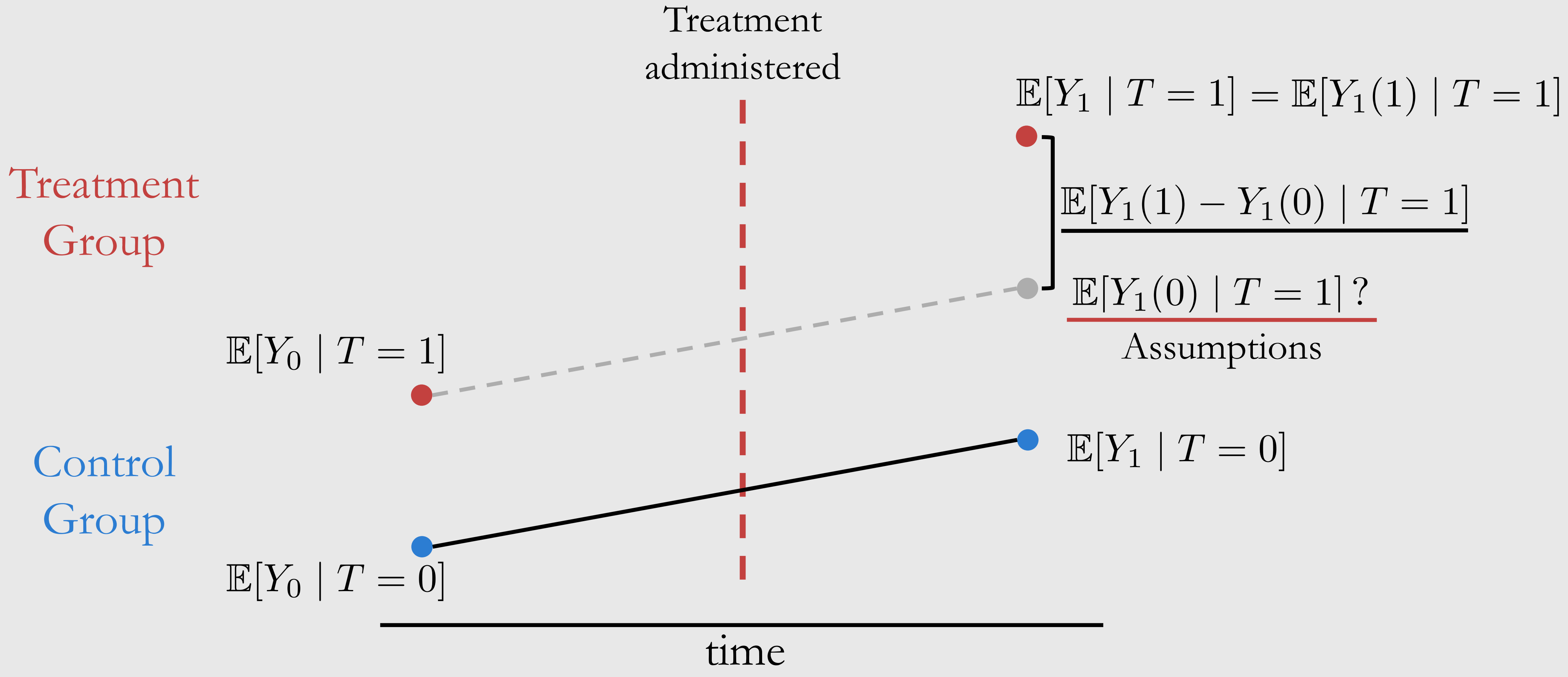
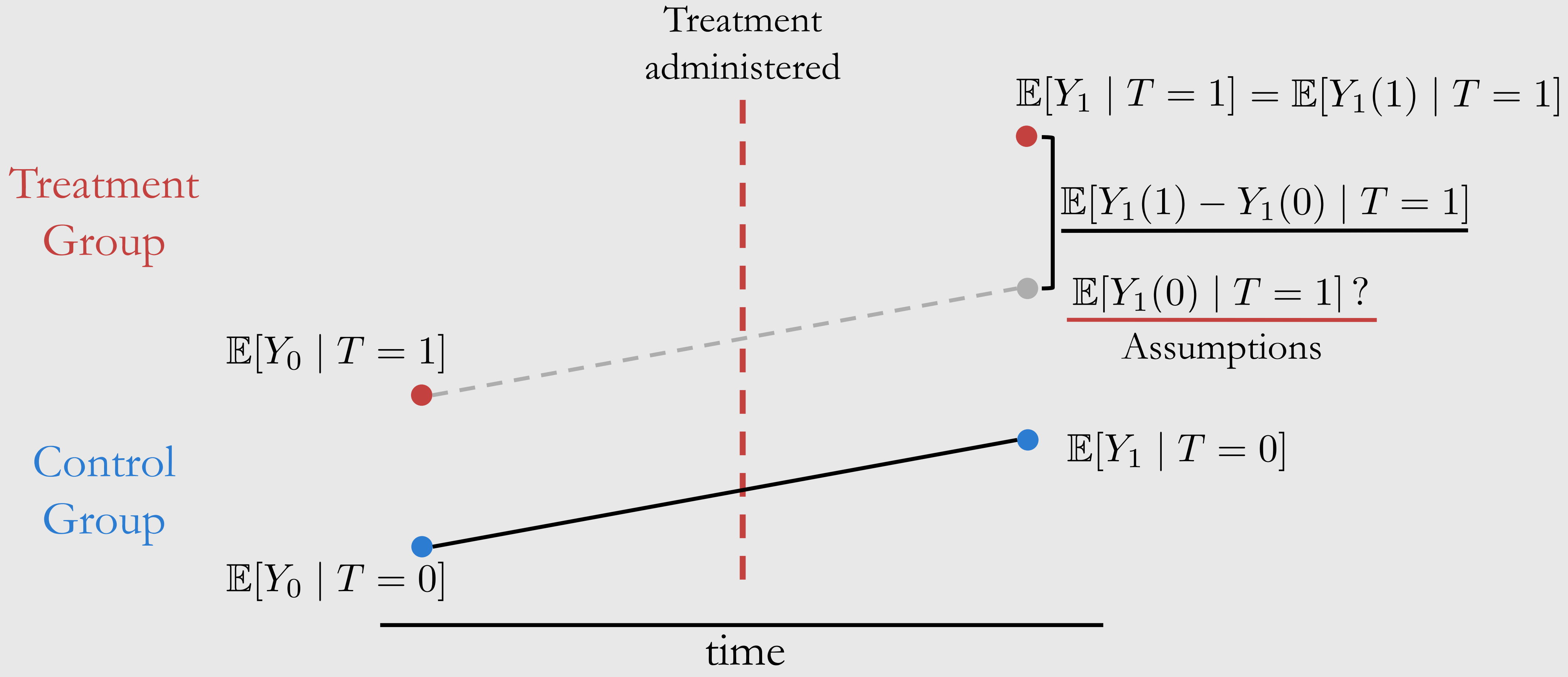


Parallel Trends Assumption: $\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0]$

No Pretreatment Effect



No Pretreatment Effect



No Pretreatment Effect Assumption: $\mathbb{E}[Y_0(1) \mid T = 1] - \mathbb{E}[Y_0(0) \mid T = 1] = 0$

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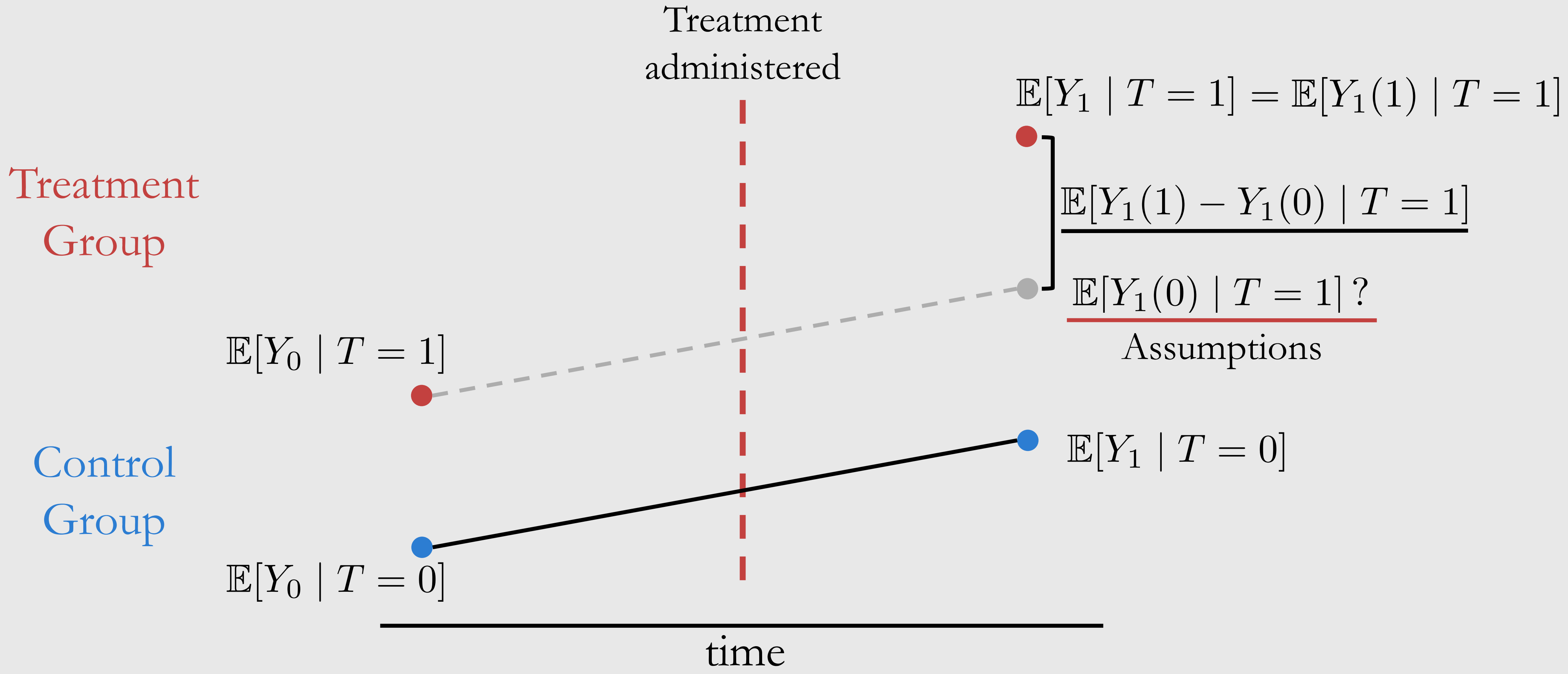

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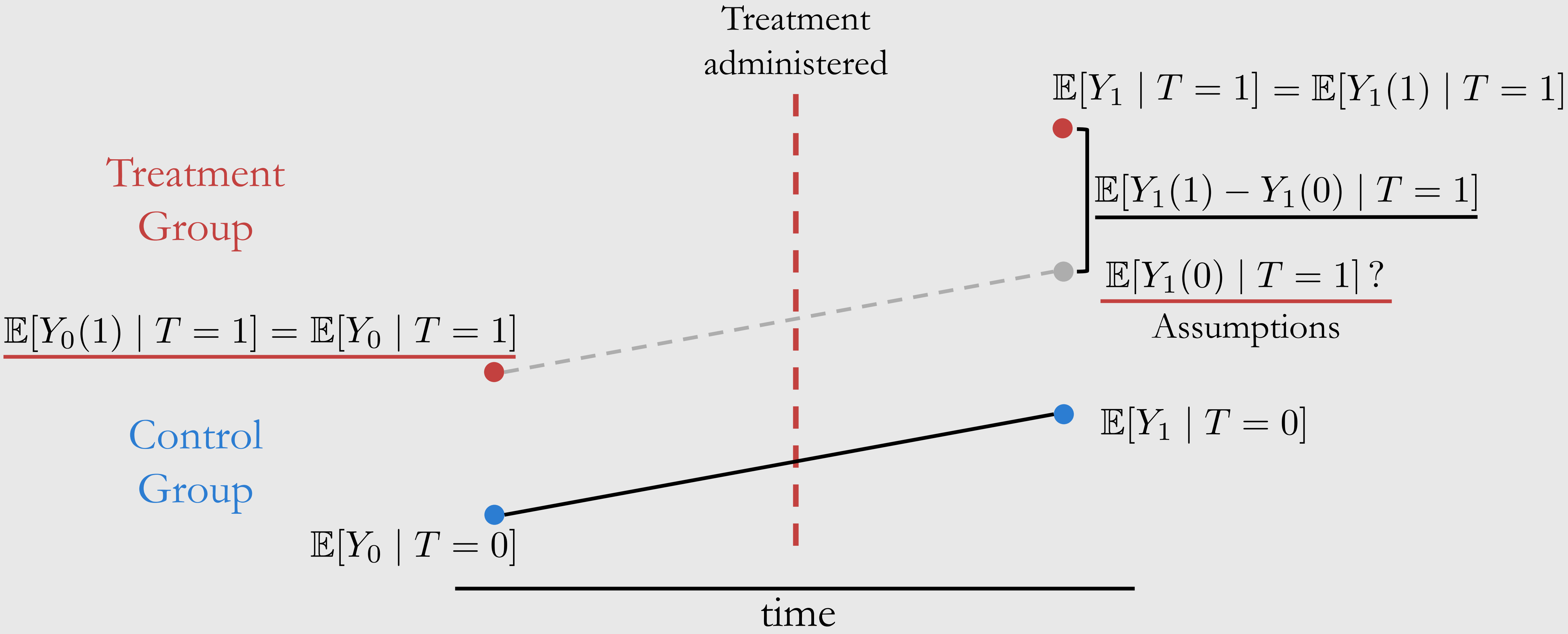
No Pretreatment Effect



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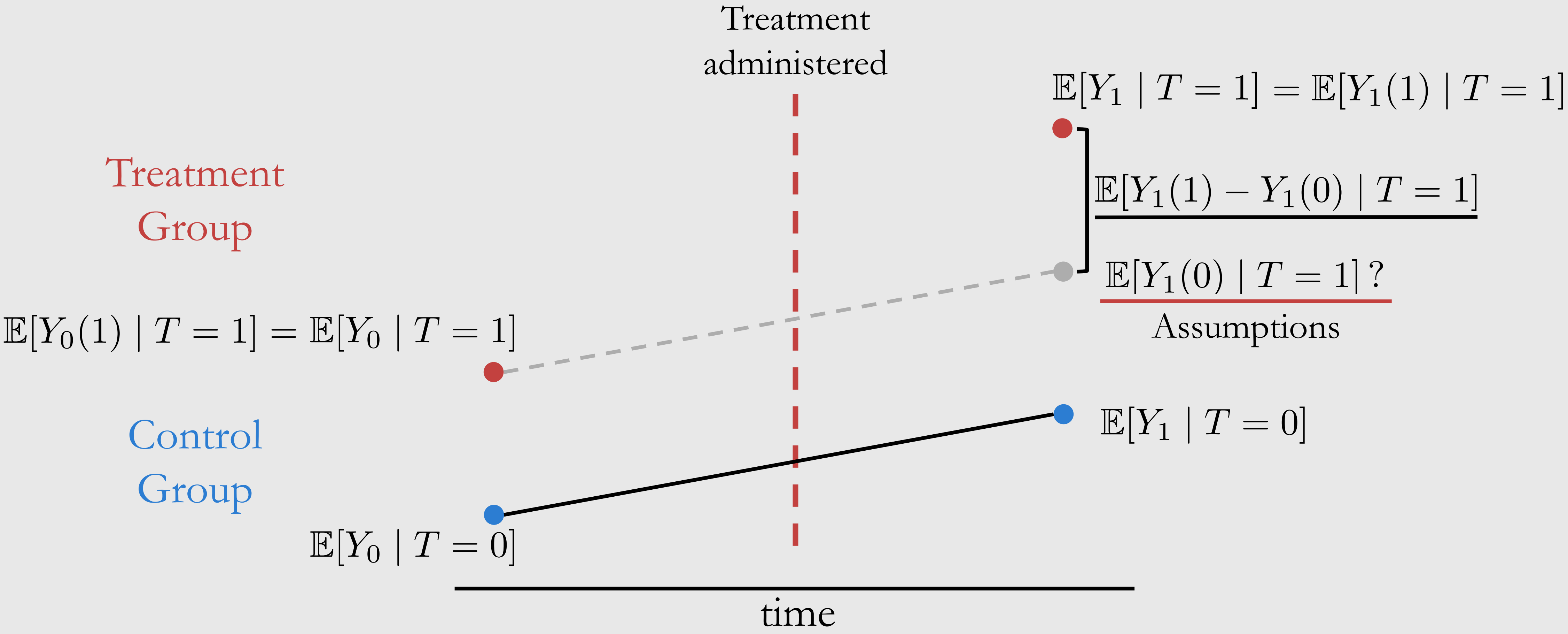

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Assumptions Change

Assumptions Change

Causal Estimand | Assumptions

Assumptions Change

Causal Estimand

$$\mathbb{E}[Y(1) - Y(0) \mid T = 1]$$

Assumptions

Assumptions Change

Causal Estimand

$$\mathbb{E}[Y(1) - Y(0) \mid T = 1]$$

Assumptions

Unconfoundedness:
$Y(0) \perp\!\!\!\perp T$

Assumptions Change

Causal Estimand

$$\mathbb{E}[Y(1) - Y(0) \mid T = 1]$$

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1]$$

Assumptions

Unconfoundedness:

$$Y(0) \perp\!\!\!\perp T$$

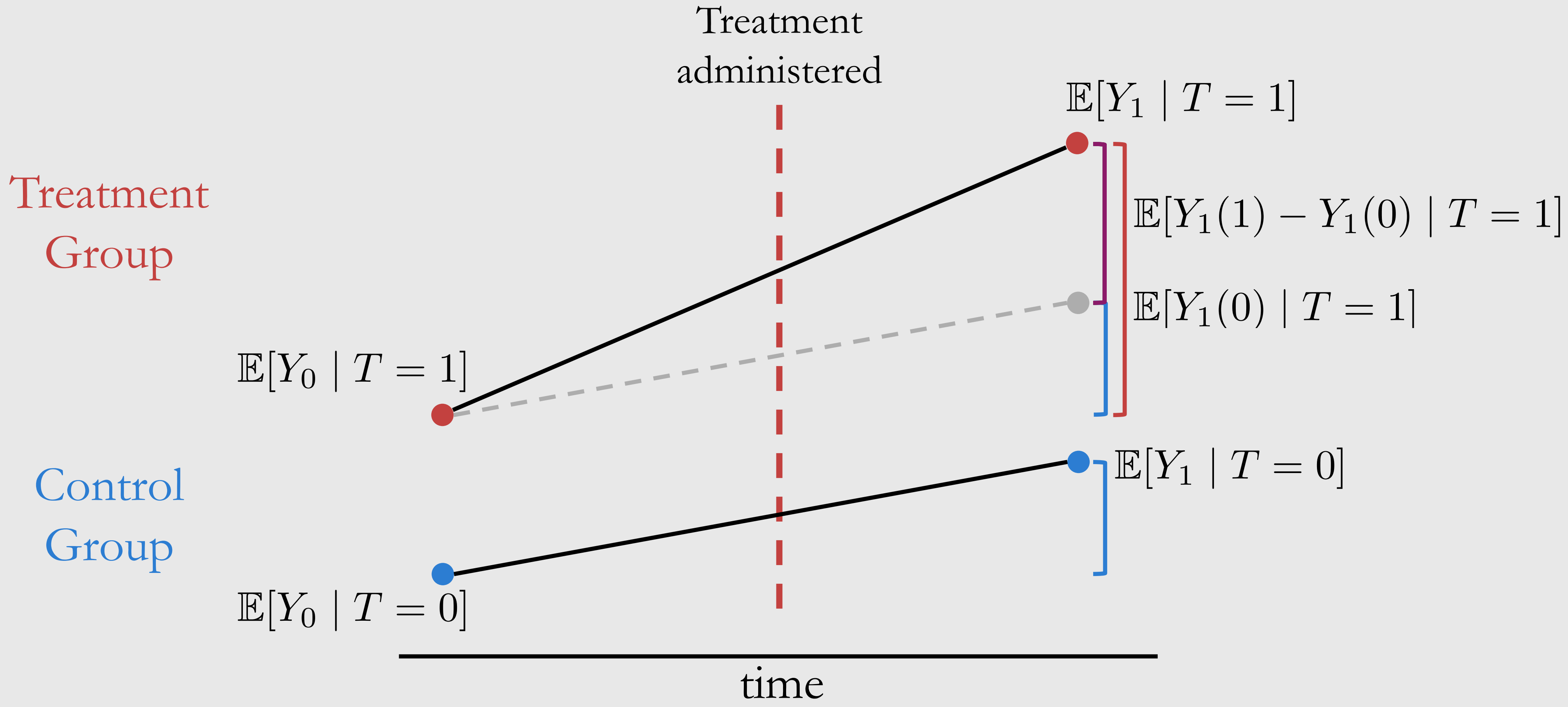
Assumptions Change

Causal Estimand	Assumptions
$\mathbb{E}[Y(1) - Y(0) \mid T = 1]$	Unconfoundedness: $Y(0) \perp\!\!\!\perp T$
$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1]$	Parallel Trends: $\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0]$

Assumptions Change

Causal Estimand	Assumptions
$\mathbb{E}[Y(1) - Y(0) \mid T = 1]$	Unconfoundedness: $Y(0) \perp\!\!\!\perp T$
$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1]$	Parallel Trends: $\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0]$ $(Y_1(0) - Y_0(0)) \perp\!\!\!\perp T$

Difference-in-Differences



$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = (\mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_0 \mid T = 1]) - (\mathbb{E}[Y_1 \mid T = 0] - \mathbb{E}[Y_0 \mid T = 0])$$

Proof

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1]$$

$$= (\mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_0 \mid T = 1]) - (\mathbb{E}[Y_1 \mid T = 0] - \mathbb{E}[Y_0 \mid T = 0])$$

Proof

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]$$

Proof

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]$$
$$= \mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]$$

Proof

$$\begin{aligned}\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] &= \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1] \\ &= \mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]\end{aligned}$$

Common trends:

$$\mathbb{E}[Y_1(0) \mid T = 1] - \mathbb{E}[Y_0(0) \mid T = 1] = \mathbb{E}[Y_1(0) \mid T = 0] - \mathbb{E}[Y_0(0) \mid T = 0]$$

Proof

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]$$
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$$\mathbb{E}[Y_1(0) \mid T = 1] = \mathbb{E}[Y_0(0) \mid T = 1] + \mathbb{E}[Y_1(0) \mid T = 0] - \mathbb{E}[Y_0(0) \mid T = 0]$$

Proof

$$\begin{aligned}\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] &= \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1] \\ &= \mathbb{E}[Y_1 \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]\end{aligned}$$

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Proof

$$\mathbb{E}[Y_1(1) - Y_1(0) \mid T = 1] = \mathbb{E}[Y_1(1) \mid T = 1] - \mathbb{E}[Y_1(0) \mid T = 1]$$
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No pretreatment effect:
$$\mathbb{E}[Y_0(1) \mid T = 1] - \mathbb{E}[Y_0(0) \mid T = 1] = 0$$

Proof

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Proof

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Question:
What is the graphical intuition behind the the various parts of the proof and assumptions on the previous slide?

Motivation and Preliminaries

Difference-in-Differences Overview

Assumptions and Proof

Problems with Difference-in-Differences

Violations of Parallel Trends

Violation: $\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1] \neq \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0]$

Violations of Parallel Trends

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Control for relevant confounders:

$$\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1, W] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0, W]$$

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Control for relevant confounders:

$$\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1, W] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0, W]$$

Violation whenever there is an interaction term between treatment and time in the structural equation for the outcome Y:

$$Y := \ldots + T\tau \quad \Longrightarrow \quad \text{Parallel trends violation}$$

Question:
When we condition on W to get parallel trends (below), what additional assumption do we need to satisfy?

$$\mathbb{E}[Y_1(0) - Y_0(0) \mid T = 1, W] = \mathbb{E}[Y_1(0) - Y_0(0) \mid T = 0, W]$$

Parallel Trends is Scale-Specific

$$\mathbb{E}[Y_1(0) \mid T = 1] - \mathbb{E}[Y_0(0) \mid T = 1] = \mathbb{E}[Y_1(0) \mid T = 0] - \mathbb{E}[Y_0(0) \mid T = 0]$$

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Does **not** imply (and is not implied by)

$$\mathbb{E}[\log Y_1(0) \mid T = 1] - \mathbb{E}[\log Y_0(0) \mid T = 1] = \mathbb{E}[\log Y_1(0) \mid T = 0] - \mathbb{E}[\log Y_0(0) \mid T = 0]$$

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This means that the parallel trends assumptions isn't nonparametric

Question:

1. Is parallel trends satisfied if time and treatment interact in producing the outcome?
2. If parallel trends is satisfied, is it also satisfied for arbitrary transformations of the outcome variable?